


Analytic Monads as ω -operads

(joint w/ Kock and Haugsong)

An operad encodes algebraic structure.
("finitary", "free resolution").

Take some base category \mathcal{Q}
(\mathcal{Q} can be sets, ∞ -cat of spaces...)

$$A = P_* P^* \quad \text{Alg}_A \simeq \mathcal{Q} \begin{array}{c} \xrightarrow{P_*} \\ \xleftarrow{P^*} \end{array} \mathcal{Q} .$$

(simplified) monadic P^* functor

Lurie's version of Barr-Beck thm:

The functor $P_*: \mathcal{Q} \rightarrow \mathcal{A}$ is monadic

if \rightarrow \bullet P_* preserves geometric realizations

"free resolution" \bullet P_* is conservative

If additionally, we want "finiteness",
then we can also ask P_* preserves
filtered colims. (filtered + realizations
 \Rightarrow sifted)

Def'n: A category \mathcal{C} is sifted if the diagonal $\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ is cofinal.

Really, we're interested in sifted colimit preserving monads $A: \mathcal{C} \rightarrow \mathcal{C}$.

Remark: \mathcal{C} is the sifted colimit completion of Fin , the category of finite sets.

Remark: This is the starting point for the Clausen-Scholze approach to w -cat of spaces, "animated sets".

Next step: try to understand w -operads as "analytic monads".

What is an operad, more specifically?

It is (in one-object case) an algebra object in symmetric sequences

$\text{Fun}(\coprod_n \mathbb{B}\Sigma_n, \mathcal{C}) \cong \prod \text{Fun}(\mathbb{B}\Sigma_n, \mathcal{C})$
under "composition product".

$\{\text{Fin}\} \uparrow$
a space with a Σ_n -action.

We'd rather view operads as certain types of monads: the main example: $\text{Sym}(X) = \text{Exp}(X) = \coprod_n X^n / \Sigma'_n \in \mathcal{Q}$.

(really the free functor, actually the monad, associated to the operad).

This monad evidently preserves sifted colimits, because, it is a composition of functors which do $(X \mapsto X^n)$.

For the same reason, the ^{endo} functor associated to a symmetric sequence $F(X) = \coprod_n (F_n \times X^n) / \Sigma'_n \rightarrow \text{Exp}(X)$ also preserves sifted colimits

sym terminal
 seq seq $F_n = \text{pt.}$
 Σ'_n

A theorem of Lurie, identifies operads with monads A equipped with a "cartesian" natural transformation $A \rightarrow \text{Exp}$.

Here, cartesian means

$$\begin{array}{ccc}
 \text{fib} \rightarrow \coprod_n A_n(x) = A(x) & \xrightarrow{\text{pull}} & \text{Exp}(x) = \coprod_n X^n / \Sigma_n \\
 \parallel & & \parallel \\
 \text{fib} \rightarrow \coprod_n A_n = A(\text{pt}) & \xrightarrow{\text{pull}} & \text{Exp}(\text{pt}) = \coprod_n B\Sigma_n
 \end{array}$$

$X^n \downarrow$ $X^n \downarrow$
 Σ_n Σ_n

Notice:

$$\begin{array}{ccccc}
 F_n & \xrightarrow{\text{fib}} & A_n & \xrightarrow{\text{fib}} & B\Sigma_n \\
 & & \parallel & & \cup \\
 & & F_n / \Sigma_n & & \text{pt}
 \end{array}$$

$$\Rightarrow A(x) = \coprod_n (F_n \times X^n) / \Sigma_n$$

where $F_n \cup \Sigma_n$ and \curvearrowright diagonal action.

Thm: (Leinster, GHK) A monad is a monad A equipped with a cartesian natural trans $A \rightarrow \text{Exp}$.
 $(\Leftrightarrow A$ preserves sifted colims).

Forgetting the monad structure and only considering underlying endofunctor,

$$\text{Fun}^{\text{an}}(\mathcal{A}, \mathcal{A})_{/\text{Exp}}^{\text{cart}} \simeq \text{Fun}(\text{Fin}, \mathcal{A})$$

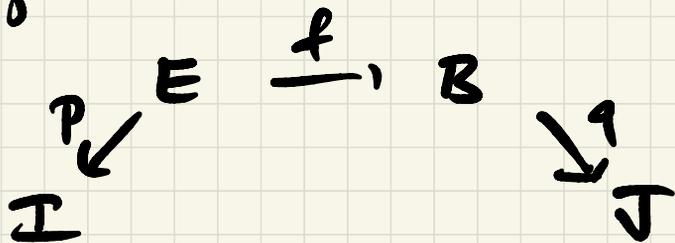
$$\simeq \mathcal{A}_{/\text{Fin}}$$

$$\simeq \mathcal{A}_{/\mathbb{N}} \cong \mathbb{B}\Sigma_n$$

so there's a

composition monoidal structure on sym seq.

Let's also forget about analyticity for the moment, and consider arbitrary polynomials:



the associated polynomial functor

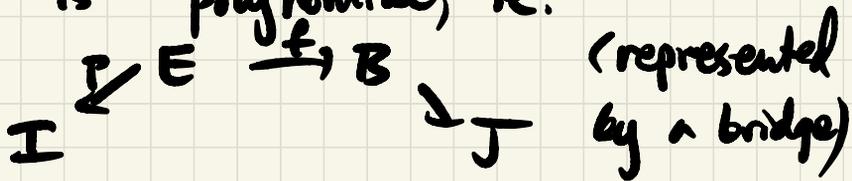
is $F: \mathcal{A}_{/I} \rightarrow \mathcal{A}_{/J}$ given by

$$F(x) = q! \cdot f_* \cdot p^* x$$

↑ dependent sum ↑ dependent product ↑ pullback

Thm. (GHK) The following are equivalent for a functor $F: \mathcal{A}_{/I} \rightarrow \mathcal{A}_{/J}$:

(1) F is polynomial, i.e.



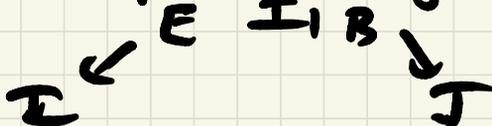
(2) F is accessible (preserves K -filt. colims for $K \gg 0$) and preserves weakly contractible limits (equivalently, "conical" limits: $\varinjlim \mathcal{C}$).

(3) F is a local right adjoint.

Thm: (GHK) The following are equiv for F : $\mathcal{L}_{I,E} \dashv \mathcal{L}_{I,J}$:

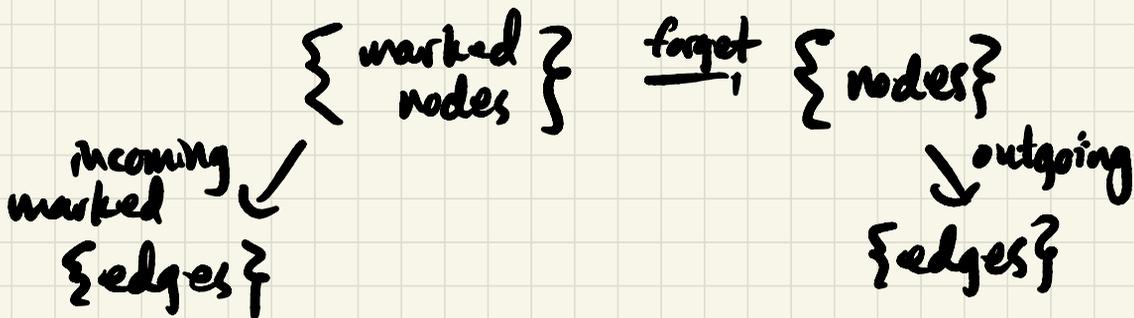
(1) F is poly and admits a map $F \rightarrow \text{Exp}$.

(2) F is represented by a bridge



(3) F preserves sifted colims and weakly contractible limits (or conical).

Runk: (Kock 2011) A tree gives rise to an analytic endofunctor:



We can reconstruct using this the Moerdijk-Weits dendroidal indexing cat:

$$\mathcal{D} \subset \text{Mnd}^{\text{an}} \xleftarrow{\text{free}} \text{End}^{\text{an}} \supset \text{Trees}$$

Thm: (G-HK) The (restricted) Yoneda embedding

$$\text{Mnd}^{\text{an}} \longrightarrow \mathcal{P}(\mathcal{D}) = \text{Fun}(\mathcal{D}^{\text{op}}, \mathcal{S})$$

is fully faithful w/ essential image the dendroidal "complete Segal spaces", $\mathcal{P}_{\text{seg}}(\mathcal{D})$.

Cor: (Heuts, Hinich-Moerdijk) $\mathcal{P}_{\text{seg}}(\mathcal{D}) \simeq \text{Op}_{\text{co}}$, so we can conclude that $\text{Mnd}^{\text{an}} \simeq \text{Op}_{\text{co}}$.