Analytic Monads as ∞-operads

(joint w/ Kock and Haugseng)

An operad encodes algebraic structure.
("finitary", "free resolution").

Take some base category $\mathcal{S}$
($\mathcal{S}$ can be sets, ∞-cat of spaces...)

$$A = p_* p^* \qquad \mathrm{Alg}_A \simeq \mathcal{Q} \underset{p^*}{\overset{p_*}{\rightleftarrows}} \mathcal{S} \quad .$$

(simplified) monadic functor

Lurie's version of Barr-Beck thm:

The functor $p_* : \mathcal{Q} \to \mathcal{S}$ is monadic if

- p_* preserves geometric realizations
- p_* is conservative

"free resolution"

If additionally, we want "finiteness", then we can also ask p_* preserves filtered colims. (filtered + realizations $\Rightarrow$ sifted)

Def'n: A category $\mathcal{C}$ is sifted if the diagonal $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is cofinal.

Really, we're interested in sifted colimit preserving monads $A : \mathcal{S} \to \mathcal{S}$.

Rmk: $\mathcal{S}$ is the sifted colimit completion of Fin, the category of finite sets.

Rmk: This is the starting point for the Clausen-Scholze approach to ∞-cat of spaces, "animated sets".

Next step: try to understand ∞-operads as "analytic monads".

What is an operad, more specifically?

It is (in one-object case) an algebra object in symmetric sequences

$$\mathrm{Fun}\Big(\coprod_n B\Sigma_n, \mathcal{S}\Big) \simeq \prod \mathrm{Fun}(B\Sigma_n, \mathcal{S})$$

under "composition product".

$\{F_n\}$

a space with a Σ_n-action.

We'd rather view operads as certain types of monads: the main example:

$$\mathrm{Sym}(X) = \mathrm{Exp}(X) = \coprod_n X^n/\Sigma_n \in \mathcal{S}.$$

(really the free functor, actually the monad, associated to the operad).

This monad evidently preserves sifted colimits, because, it is a composition of functors which do ($X \mapsto X^n$).

For the same reason, the endo functor associated to a symmetric sequence

$$F(X) = \coprod_n (F_n \times X^n)/\Sigma_n \longrightarrow \mathrm{Exp}(X)$$

also preserves sifted colimits

terminal sym seq $\Sigma_n \circlearrowright F_n = pt.$

A theorem of Leinster, identifies operads with monads A equipped with a "cartesian" natural transformation

$$A \longrightarrow \mathrm{Exp}.$$

Here, cartesian means

$$\begin{tikzcd}
 & X^n \arrow[d] & X^n \arrow[d] \\
\coprod_n A_n(X) = A(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull}"] & \mathrm{Exp}(X) = \coprod_n X^n/\Sigma_n \arrow[d] & {} \arrow[d] \\
\coprod_n A_n = A(pt) \arrow[r] & \mathrm{Exp}(pt) = \coprod_n B\Sigma_n & {}
\end{tikzcd}$$

$\text{fib} \to \coprod_n A_n(X)$, $\;\|\;$, $\text{fib} \to \coprod_n A_n$

Notice:

$$\begin{tikzcd}
F_n \circlearrowleft \Sigma_n \arrow[r, "\text{fib}"] & A_n \arrow[r] \arrow[d, equal] & B\Sigma_n \\
 & F_n/\Sigma_n & pt \arrow[u]
\end{tikzcd}$$

$$\Rightarrow A(X) = \coprod_n (F_n \times X^n)/\Sigma_n$$

where $F_n \circlearrowleft \Sigma_n$ and $\circlearrowleft$ diagonal action.

<u>Thm</u>: (Leinster, GHK) An operad is a monad A equipped with a cartesian natural trans $A \to \mathrm{Exp}$.
($\Leftrightarrow$ A preserves sifted colims).

Forgetting the monad structure and only considering underlying endofunctor,

$$\mathrm{Fun}^{an}(\mathcal{S},\mathcal{S})^{cart}_{/\mathrm{Exp}} \simeq \mathrm{Fun}(\mathrm{Fin},\mathcal{S}) \simeq \mathcal{S}_{/\mathrm{Fin}} \simeq \mathcal{S}_{/\coprod_n B\Sigma_n}$$

(symmetric sequences)

so there's a composition monoidal structure on sym seq.

Let's also forget about analyticity for the moment, and consider arbitrary polynomials:

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[r, "f"] & B \arrow[dr, "q"] & \\ I & & & J \end{tikzcd}$$

the associated polynomial functor is $F : \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ given by

$$F(X) = q_! \, f_* \, p^* X$$

dependent sum   dependent product   pullback

Thm. (GHK) The following are equivalent for a functor $F : \mathcal{S}_{/I} \to \mathcal{S}_{/J}$:

(1) F is polynomial, i.e.

$$\begin{tikzcd} & E \arrow[dl, "p"'] \arrow[r, "f"] & B \arrow[dr] & \\ I & & & J \end{tikzcd}$$

(represented by a bridge)

(2) F is accessible (preserves κ-filt. colims for $\kappa \gg 0$) and preserves weakly contractible limits (equivalently, "conical" limits: $\mathcal{D}^{\triangleleft} \to \mathcal{C}$).

(3) F is a local right adjoint.

Thm: (GHK) The following are equiv for $F : \mathcal{C}_{/I} \to \mathcal{C}_{/J}$:

(1) F is poly and admits a map $F \to \mathrm{Exp}$.

(2) F is represented by a bridge

$$\begin{tikzcd} & E \arrow[dl] \arrow[r, "f"] & B \arrow[dr] & \\ I & & & J \end{tikzcd}$$

where f is étale (finite discrete fibers)

(3) F preserves sifted colims and weakly contractible lims (or conical).

<u>Rmk</u>: (Kock 2011) A tree gives rise to an analytic endofunctor:

$$
\begin{tikzcd}
\text{marked}\ \{\text{edges}\} & \left\{\begin{matrix}\text{marked} \\ \text{nodes}\end{matrix}\right\} \arrow[l, "\text{incoming}"'] \arrow[r, "\text{forget}"] & \{\text{nodes}\} \arrow[r, "\text{outgoing}"] & \{\text{edges}\}
\end{tikzcd}
$$

We can reconstruct using this the Moerdijk-Weiss dendroidal indexing cat:

$$\Omega \subset \mathrm{Mnd}^{an} \xleftarrow{\text{free}} \mathrm{End}^{an} \supset \mathrm{Trees}$$

<u>Thm</u>: (GHK) The (restricted) Yoneda embedding

$$\mathrm{Mnd}^{an} \longrightarrow \mathcal{P}(\Omega) = \mathrm{Fun}(\Omega^{op}, \mathcal{S})$$

is fully faithful w/ essential image the dendroidal "complete Segal spaces", $\mathcal{P}_{seg}(\Omega)$.

<u>Cor</u>: (Heuts-Hinich-Moerdijk) $\mathcal{P}_{seg}(\Omega) \simeq \mathrm{Op}_\infty$, so we can conclude that $\mathrm{Mnd}^{an} \simeq \mathrm{Op}_\infty$.