Relative Topology, Motion Planning, and Coverage Problems

Topos Colloquium

February 18, 2021

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Categories and Geometry/Topology

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- Emmy Noether pointed to functoriality in 1930's
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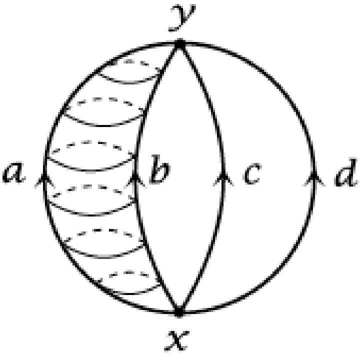
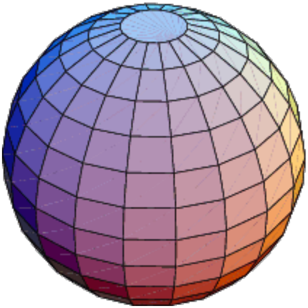
- There are dictionaries between categories and spaces - *nerve construction*
- Functoriality and categories are *the* key ideas for algebraic topology

Algebraic Topology



Homology

Algebraic Topology



$$H_0 = k, H_1 = 0, H_2 = k, H_i = 0 \text{ for } k > 2$$

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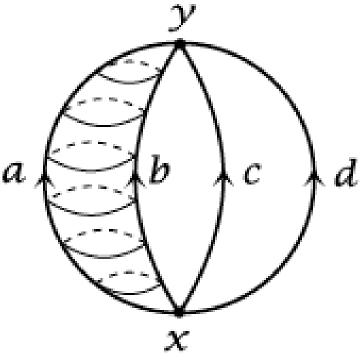

$H_0 = k, H_1 = k^3, H_2 = k, H_i = 0$ for $k > 2$

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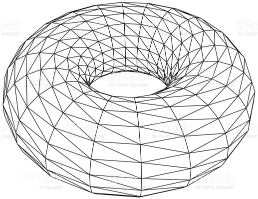

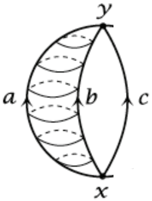

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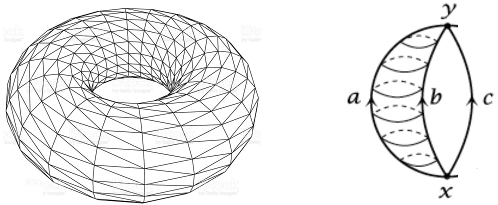

Same homology

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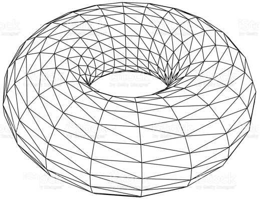

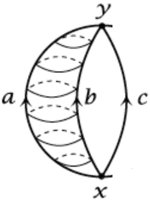

Different spaces

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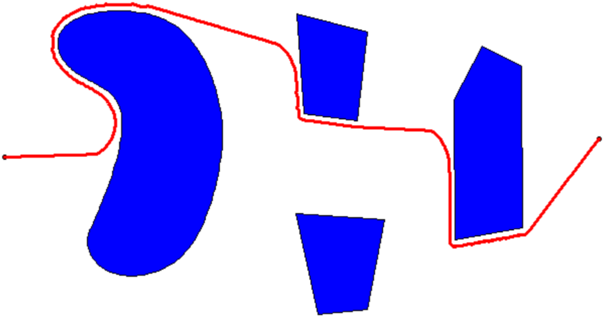
- We will be interested in the situation where $B = [0, 1]$

The Topology of Complements

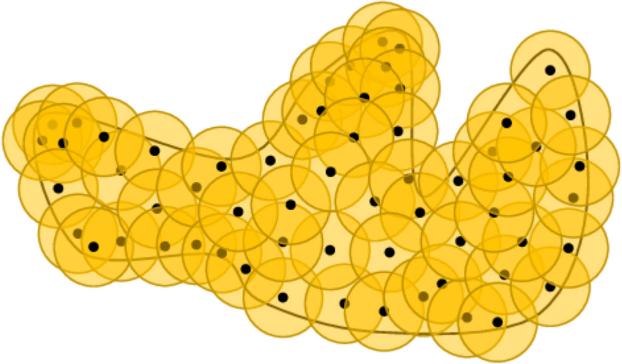
Suppose we have $Y \subseteq X$ an embedding of topological spaces, and we have topological information about X and Y. What can be said about the topology of $X - Y$?

Robotics



Motion planning through obstacles

Sensor Nets



Covered region for a sensor net

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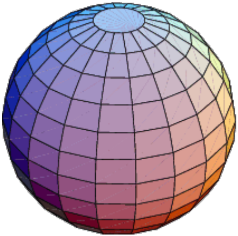
- Alexander duality theorem

Manifolds

- A space X is an n-dimensional manifold if it is locally like $\mathbb{R}^n$



Manifold

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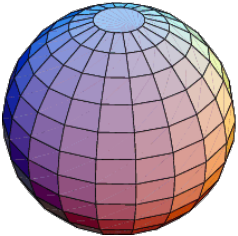
- A space X is an n-dimensional manifold if it is locally like $\mathbb{R}^n$
- Means every point has a neighborhood homeomorphic to an open disc in $\mathbb{R}^n$



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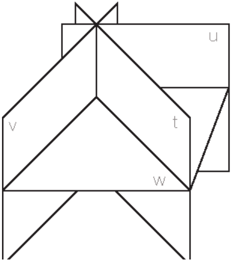

Not a manifold

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- *Contravariant*

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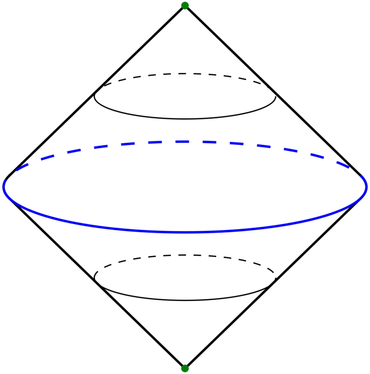
Tells us we can recover the homology or cohomology of complements. Can we recover the actual space from Y?

The Suspension of a Space



The suspension of a circle is a sphere

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- Key fact is that for large N all embeddings of Y in S^N are isotopic

Spanier-Whitehead Duality

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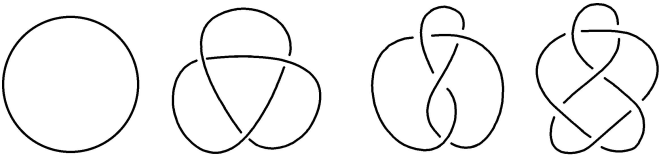
Yes

Knot Theory



Embeddings of circle in $\mathbb{R}^3$

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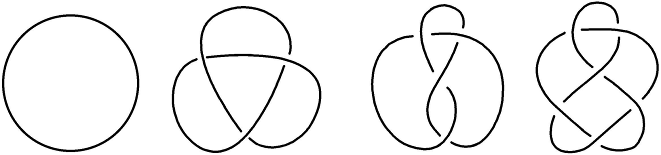
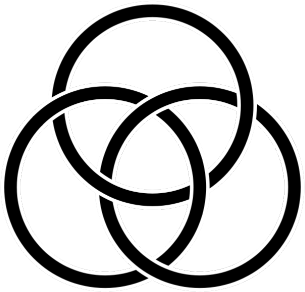


Fundamental group of knot complement is a key invariant of a knot

Link Theory



Embeddings of disjoint union of circles in $\mathbb{R}^3$

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- Can we impose additional structure on homology or cohomology which detects unstable phenomena?

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- $\mathbb{T}$ a torus, $H_0(\mathbb{T}) = k$, $H_1(\mathbb{T}) = k^2$, and $H_2(\mathbb{T}) = k$.

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- $H^*(X)$ becomes a graded k-algebra

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- Torus gives an example

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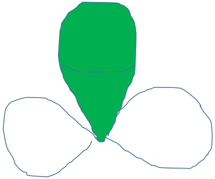
- Let $\mathcal{B}$ be the bouquet $S^1 \vee S^1 \vee S^2$



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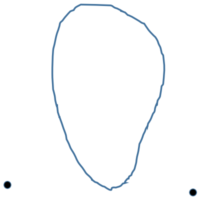

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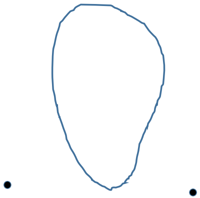

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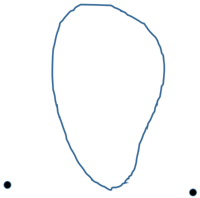

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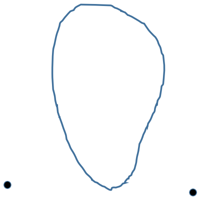

- Follows that cup products of positive elements vanish
- Homology and cohomology of $\mathcal{B}$ and $\mathbb{T}$ are identical as vector spaces
- Cup product shows them to be distinct as unstable homotopy types. On the other hand, $\Sigma\mathbb{T} \simeq \Sigma\mathcal{B}$.

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- $H_0(-)$ does not determine $\pi_0(-)$ *as a set-valued functor*

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- The set of k-algebra homomorphisms $H^0(X) \to k$ is in one to one correspondence with the elements of $\pi_0(X)$
- Means that we *can* recover π_0 from the k-algebra valued functor $H^0(-)$

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- Is it possible to use the same methods to recover cup products of the complement, and consequently the set π_0 applied to the complement?

Cup Products of Complements

- The various duality theorems allow us to understand the homology of the complements, including H_0.
- Is it possible to use the same methods to recover cup products of the complement, and consequently the set π_0 applied to the complement?
- This can be done, using the fact that cup products are induced by a map, namely the diagonal map, and a functoriality result for the duality theorems

Functoriality of Alexander Duality

All the Alexander Duality isomorphism for $X = S^n$ described above is functorial, in the sense that for an inclusion $Y_0 \subseteq Y_1 \subseteq X$, the diagram

$$\begin{array}{ccc} H^i(X - Y_0) & \longrightarrow & H_{n-i-1}(Y_0) \\ \downarrow & & \downarrow \\ H^i(X - Y_1) & \longrightarrow & H_{n-i-1}(Y_1) \end{array}$$

commutes. Note that we have $X - Y_1 \hookrightarrow X - Y_0$. There are analogous statements for $X = \mathbb{R}^n$ or more general.

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- Obtain map $C(A \times A) \to C_\Delta A$ inducing the cup product on $H^0(A)$ via functoriality of Alexander Duality

Time Varying Robotics and Sensor Net Problems

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- The time varying obstacles will now consist of a subspace $Y \subseteq X$

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- By an *admissible path*, we'll mean a section

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i.e. a continuous map $\sigma : [0,1] \to Y$ so that $\pi \circ \sigma = id_{[0,1]}$

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- What is the structure of the space of sections?

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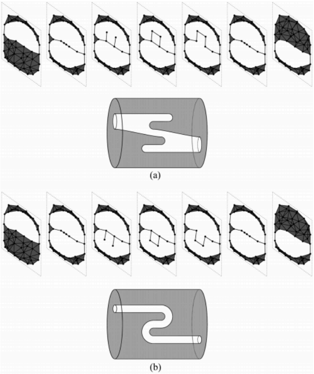

Analogue of two distinct knots in this setting (Henry Adams)

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- Related notion of *parametrized homology*, S. Kalisnik

Sheaves

A *sheaf* on a topological space with values in a category $\underline{C}$ is contravariant functor F from the category of open subsets of X to $\underline{C}$ so that for any two open sets $U, V \subseteq X$, the diagram

$$\begin{array}{ccc} F(U \cup V) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \cap V) \end{array}$$

is a pullback or an equalizer.

Sheaves - Examples

- Functions on X with values in $\mathbb{R}$ creates a sheaf of $\mathbb{R}$-vector spaces

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Sheaves

A *sheaf* on a topological space with values in a category $\underline{C}$ is covariant functor F from the category of open subsets of X to $\underline{C}$ so that for any two open sets $U, V \subseteq X$, the diagram

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- H^0 creates a sheaf of k-algebras.
- Should be able to construct the cosheaf π_0 as the cosheaf of algebra homomorphisms from H^0 to the constant sheaf k.

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- Analogue of U.A.s.s. for the setting of spaces over $[0, 1]$ could address Problem 3 above. (W. Mackey has an approach)
- Embedding calculus is a technique for parametrizing the embeddings of one manifold in another. A. Jin and G. Arone working on applying it in the setting of spaces over $[0, 1]$.