The logic of contextuality

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- We will attempt to be fairly self-contained

Background

John von Neumann, in his seminal *Mathematical Foundations of Quantum Mechanics* (German edition 1932, based on earlier papers from 1927), identified quantum **properties** or **propositions** as projectors on a Hilbert Space $\mathcal{H}$, i.e. linear operators P on $\mathcal{H}$ which are bounded, self-adjoint ($P = P^\dagger$) and idempotent ($P^2 = P$).

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A different approach was proposed by Simon Kochen and Ernst Specker (both logicians) in their seminal work on contextuality in the 1960's, based on **partial Boolean algebras**.

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- Non-locality (as in Bell's theorem) is a special case
- Highly implicated in many cases of quantum advantage
- Contextuality arises where there is a family of data which is **locally consistent but globally inconsistent**

Partial Boolean algebras

A partial Boolean algebra A is given by a set (also written A), constants 0, 1, a reflexive, symmetric binary relation $\odot$ on A, read as "commeasurability" or "compatibility", a total unary operation $\neg$, and partial binary operations $\wedge$, $\vee$ with domain $\odot$.

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Morphisms of partial Boolean operations are maps preserving commeasurability, and the operations wherever defined. This gives a category **pBA**.

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We shall call this the **K-S property** of a pBA.

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Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S.*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$, $A \models \varphi(\vec{a})$.

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How can the world be this way? Still an ongoing debate, an enduring mystery ...

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Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

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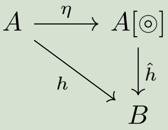
More generally, we use this approach to prove the following result, which freely generates from a given partial Boolean algebra a new one where prescribed additional commeasurability relations are enforced between its elements.

Theorem

Given a partial Boolean algebra A and a binary relation $\circledcirc$ on A, there is a partial Boolean algebra $A[\circledcirc]$ such that:

- *There is a* **pBA**-*morphism* $\eta \colon A \longrightarrow A[\circledcirc]$ *such that* $a \circledcirc b \;\Rightarrow\; \eta(a) \odot_{A[\circledcirc]} \eta(b)$.
- *For every partial Boolean algebra B and* **pBA**-*morphism* $h \colon A \longrightarrow B$ *such that* $a \circledcirc b \;\Rightarrow\; h(a) \odot_B h(b)$, *there is a unique homomorphism* $\hat{h} \colon A[\circledcirc] \longrightarrow B$ *such that*



This result is proved constructively, by giving proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A. (In fact, we start with a set of "pre-terms", and also give rules for definedness).

The inductive construction

$$\frac{a \in A}{\imath(a)\downarrow} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

$$\frac{}{0 \equiv \imath(0_A),\ 1 \equiv \imath(1_A),\ \neg\imath(a) \equiv \imath(\neg_A a)}$$

$$\frac{a \odot_A b}{\imath(a) \wedge \imath(b) \equiv \imath(a \wedge_A b),\ \imath(a) \vee \imath(b) \equiv \imath(a \vee_A b)}$$

$$\frac{}{0\downarrow,\ 1\downarrow} \qquad \frac{t \odot u}{t \wedge u\downarrow,\ t \vee u\downarrow} \qquad \frac{t\downarrow}{\neg t\downarrow}$$

$$\frac{t\downarrow}{t \odot t,\ t \odot 0,\ t \odot 1} \qquad \frac{t \odot u}{u \odot t} \qquad \frac{t \odot u,\ t \odot v,\ u \odot v}{t \wedge u \odot v,\ t \vee u \odot v} \qquad \frac{t \odot u}{\neg t \odot u}$$

$$\frac{t\downarrow}{t \equiv t} \qquad \frac{t \equiv u}{u \equiv v} \qquad \frac{t \equiv u,\ u \equiv v}{t \equiv v} \qquad \frac{t \equiv u,\ u \odot v}{t \odot v}$$

$$\frac{\varphi(\vec{x}) \equiv_{\mathsf{Bool}} \psi(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{\varphi(\vec{v}) \equiv \psi(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \circledcirc a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\circledcirc, \equiv]$.

Theorem

Let $h \colon A \longrightarrow B$ be a **pBA***-morphism such that $a \circledcirc a' \;\Rightarrow\; h(a) = h(a')$. Then there is a unique* **pBA***-morphism $\hat{h} \colon A[\circledcirc, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This result can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

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To resolve the apparent contradiction, note that **BA** is an equational variety of algebras over **Set**.

As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

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In the case of a partial Boolean algebra with the K-S property of not having a homomorphism to **2**, the colimit of its diagram of boolean subalgebras must be **1**.

KS-property and colimits

We can turn this into a theorem:

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Let A be a partial Boolean algebra. The following are equivalent:

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A partial Boolean algebra with the K-S property – such as $\mathsf{P}(\mathcal{H})$ – holds this implicitly contradictory information together in a single structure.

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Conversely, suppose that $A[A^2] = \mathbf{1}$, and there is a morphism $A \longrightarrow B$ to a Boolean algebra A. By the universal property of $A[A^2]$, there is a morphism $A[A^2] \longrightarrow B$, and since $A[A^2] = \mathbf{1}$, we must have $B = \mathbf{1}$. Thus A is K-S. □

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One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

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Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** such that the functor $\mathsf{P} \colon \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ is **strong monoidal** with respect to this structure, i.e. such that $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$?

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A positive answer to this question would offer a complete logical characterisation of the Hilbert space tensor product, and provide an important step towards giving logical foundations for quantum theory in a form useful for quantum information and computation.

Tensor products of partial Boolean algebras

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In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

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Our Theorem 2 allows us to give an explicit description of this construction using generators and relations.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

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There is a lax monoidal functor $\mathsf{P}\colon \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ into $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

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It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

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Interestingly, in (Kochen 2015) it is shown that the images of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$, for any finite-dimensional $\mathcal{H}$ and $\mathcal{K}$, generate $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$. This is used there to justify the claim contradicted by the previous paragraph. The gap in the argument is that more relations hold in $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ than in $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K})$.

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Nevertheless, this result is very suggestive. It poses the challenge of finding a stronger notion of tensor product.

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An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t{\downarrow}$ can be derived, then $u{\downarrow}$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

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To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

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However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not.

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Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

Logical exclusivity principle

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Thus $a \perp b$ is a weaker requirement than $a \wedge b = 0$, although the two would be equivalent in a Boolean algebra. The point is that, in a general partial Boolean algebra, one might have exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

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Definition

A partial Boolean algebra A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.
We write **epBA** for the full subcategory of **pBA** whose objects are partial Boolean algebras satisfying LEP.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

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As an immediate consequence, any $\mathsf{P}(\mathcal{H})$ satisfies LEP.

A reflective adjunction for logical exclusivity

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However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

This LEP-isation is analogous to e.g. the way one can 'abelianise' any group, or use Stone–Čech compactification to form a compact Hausdorff space from any topological space.

Theorem

The category **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I\colon \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X\colon \mathbf{pBA} \longrightarrow \mathbf{epBA}$. *Concretely, to any partial Boolean algebra* A, *we can associate a Boolean algebra* $X(A) = A[\bot]^*$ *which satisfies LEP such that:*

- *there is a homomorphism* $\eta\colon A \longrightarrow A[\bot]^*$;
- *for any homomorphism* $h\colon A \longrightarrow B$ *where* B *is a partial Boolean algebra* B *satisfying LEP, there is a unique homomorphism* $\hat{h}\colon A[\bot]^* \longrightarrow B$ *such that:*

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The proof of this result follows from a simple adaptation of the proof of Theorem 2, namely adding the following rule to the inductive system presented in Table 1:

$$\frac{u \wedge t \equiv u, \ v \wedge \neg t \equiv v}{u \odot v}$$

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This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

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How close does it it get us to the full Hilbert space tensor product?

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Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

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Theorem (K-S faithfulness of extensions)

Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

Proof.

If A is not K-S, it has a homomorphism to a non-trivial Boolean algebra B. By the universal property of $A[R]$, there is a homomorphism $\hat{h} : A[R] \longrightarrow B$. Thus $A[R]$ is not K-S. Conversely, if there is a morphism $k : A[R] \longrightarrow B$ to a non-trivial Boolean algebra B, then $k \circ \eta : A \longrightarrow B$, so A is not K-S. □

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If A and B are not K-S, they have homomorphisms to $\mathbf{2}$, and hence so does $A \oplus B$. Applying the previous theorem inductively $k+1$ times, so does $A \otimes B[\perp]^k = A \oplus B[\oplus][\perp]^k$. □

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Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a K-S paradox if none was present if A or B.

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So we have narrowed, but not closed the gap ...

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- Characterizing the contradictions which can be realized in QM

Higher-dimensional contextuality

We recall the following quotation from Ernst Specker given by Cabello:

> *Do you know what, according to me, is the fundamental theorem of quantum mechanics? ... That is, if you have several questions and you can answer any two of them, then you can also answer all three of them. This seems to me very fundamental.*

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This refers to the **binarity** of compatibility in quantum mechanics. A set of observables is compatible if they are pairwise so. This is built in to the definition of partial Boolean algebras, wit the binary relation of compatibility.

Higher-dimensional contextuality

We recall the following quotation from Ernst Specker given by Cabello:

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This refers to the **binarity** of compatibility in quantum mechanics. A set of observables is compatible if they are pairwise so. This is built in to the definition of partial Boolean algebras, wit the binary relation of compatibility.

However, in the general theory of contextuality, as developed e.g. in the sheaf-theoretic approach, more general forms of compatibility are considered, represented by simplicial complexes. Can partial Boolean algebras be adapted to this more general format, and how much of the theory carries over?

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Is there a "logical" proof?