Quotient completions

for

topos-like structures

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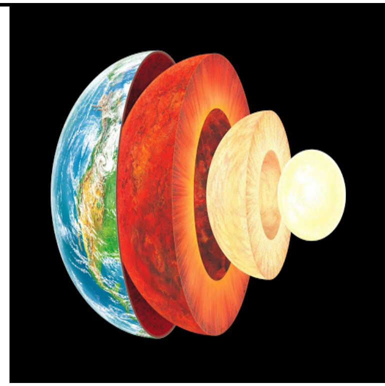
Abstract of our talk



Motivations

Unifying Exact completions as completions of doctrines

Applications to the Tripos-to -Topos construction

Elementary quotient completion

Applications to quasi-toposes and to predicative toposes

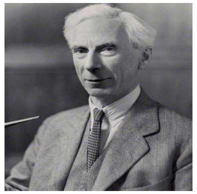
Our goal

find a "**topos**" (= **structure** of typical place)

where to model predicative constructive mathematics

à la **Poincarè-Weyl-Feferman**

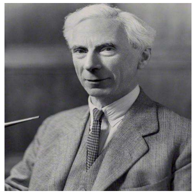
Characteristics of predicative definitions

in the sense of **Russell** 

"Whatever involves an apparent variable must *not be among the possible values* of that variable."

impredicativity of topos internal language

The **generic internal language** of elementary toposes

is impredicative in the sense of **Russell**  and **Poincarè** 

for the presence of the subobject classifier/ power-objects

because

for any formula $\phi(x, U)$

$$\{x \in Nat \mid \forall U \in \mathcal{P}(Nat)\, \phi(x, U)\} \,\varepsilon\, \mathcal{P}(Nat)$$

the power-object $\mathcal{P}(Nat)$ is closed under a subset

defined by a quantification over all subsets

including itself!

classical predicative mathematics is viable

according to **Hermann Weyl** 

"... **the continuum**... *cannot at all be battered into a single set of elements*".

⇒ following **Poincarè** 

"*only predicative definitions can be accepted on* **infinite classes**"

also confirmed by **Friedman -Simpson**'s program of reverse mathematics:

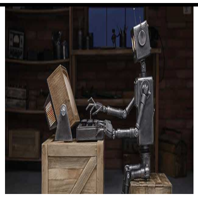
"*most basic classical mathematics* can be formalized in a **predicative** foundation

à la **Feferman** 

is constructive predicative mathematics viable ??

under investigation...!

What is **constructive mathematics** *?*



constructive mathematics =maths with a computational interpretation

such that

CONSTRUCTIVE functions =COMPUTABLE functions

CONSTRUCTIVE proofs= some programs

including a extraction of a computable **witness**

from proven existential statements

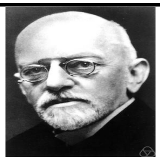
Example of a constructive foundation à la Poincarè, Weyl and Feferman

our **Minimalist Foundation**

as a two-level theory

from [Maietti'09] in agreement with [M. Sambin2005]

the **Minimalist Foundation** is an answer to

revised Hilbert program: 

need of a **trustable** foundation for **mathematics**

predicative à la à la **Poincarè**, **Weyl**, **Feferman**   

constructive à la **Bishop**

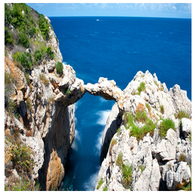
open-ended to further extensions according to **Martin-Löf**

for **computed-aided formalization of its proofs** as advocated by **V. Voevodsky**

and **compatible** with **most relevant foundations**

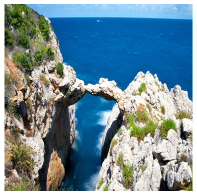
our current notion of constructive foundation for mathematics

j.w.w. G. Sambin

two-level theory 	extensional level (used by mathematicians to developed their proofs) ⇓ interpreted via a **QUOTIENT model** intensional level (language suitable for computer-aided formalization of proofs)
extra level	⇓ realizability level (used by computer scientists to extract programs) validating axiom of choice + formal Church's thesis

The **two- level** **Minimalist Foundation MF**

ideated with G. Sambin in 2005, completed in [M2009]

<table>
<tr><td>extensional level:
</td><td>eMF (proof-irrelevant local set theory of predicative quasi-toposes)

$\Downarrow$ via a quotient model</td></tr>
<tr><td>intensional level:</td><td>iMF (proof-relevant predicative intensional type theory)</td></tr>
</table>

Plurality of foundations has a Minimalist Foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of toposes Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory HoTT and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the MINIMALIST FOUNDATION (**MF**) is a common core

our categorical tool: Lawvere's doctrines

we employ **Lawvere's doctrines**

to produce **topos-like** constructions via **quotient-completions**

with applications to **Categorical Model Theory**

of **foundations** of **predicative** and **constructive mathematics**

final aim: to establish **independence proof results**

Key applications of quotient completions of doctrines in foundation of mathematics

to model extensional constructions

including quotient sets

with undecidable equalities

$\downarrow$ in

an INTENSIONAL theory with decidable equalities

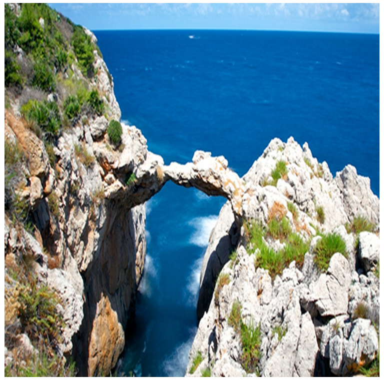
NOT closed under quotient sets!!!

=RELIABLE base for a proof-assistant

(like Swedish **Agda**/ French **COQ**)

final aim:

Extraction of PROGRAMS from CONSTRUCTIVE proofs



exact completions relative to doctrines

with G. Rosolini

in ”Unifying exact completion” APCS, 2015

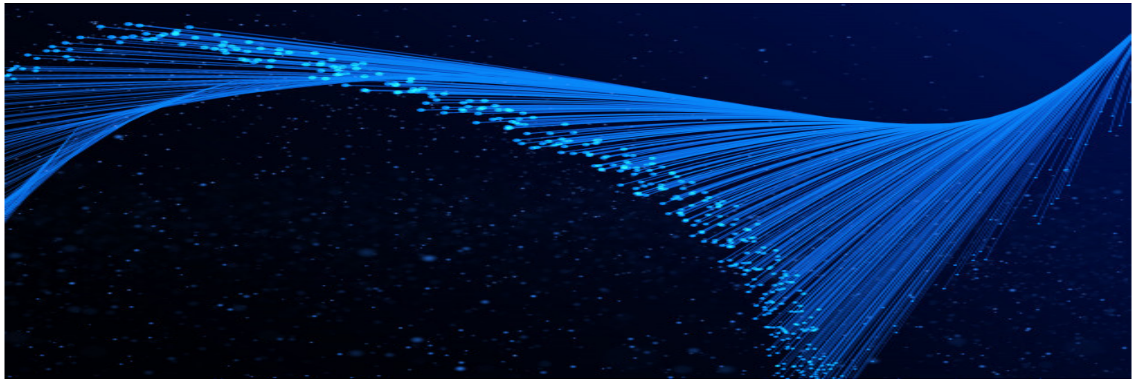
we *presented* well known **exact completions**

(of weakly lef exact finite product cats/of regular cats) [Carboni-Vitale’98]:

by *relativizing* it to an existential elementary doctrine

$$\mathbf{P}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$$

(and NOT just to a category!)



inspiring example of doctrinal exact completions

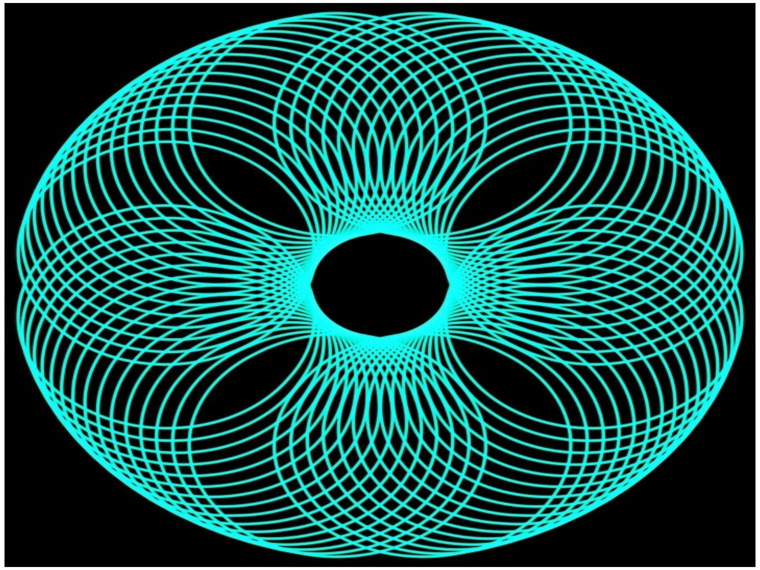
with G. Rosolini

[J. M. E. Hyland, P. T. Johnstone, A. M. Pitts.'80]

the **tripos-to-topos** construction

of the **topos** $\mathscr{T}_P$

from a **tripos** $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSL}$



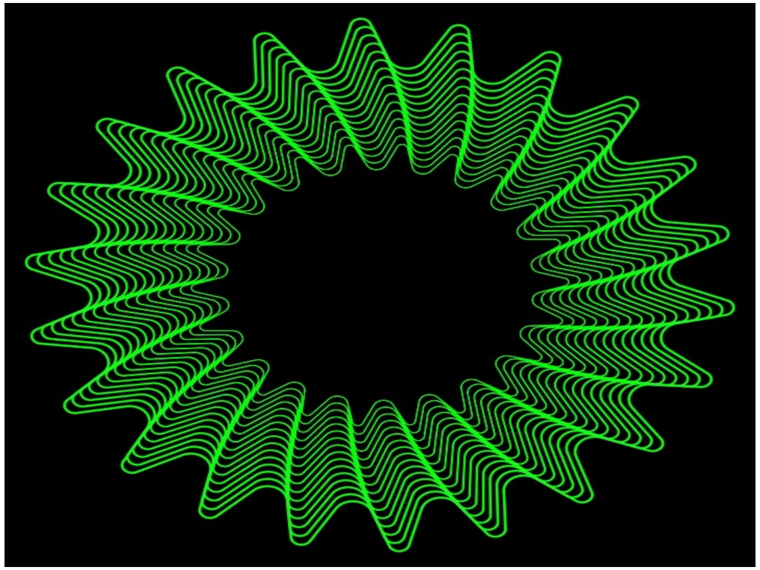
idea of quotient completion

completion of a category $\mathcal{C}$ with quotients

relative to a **Lawvere's** elementary DOCTRINE on $\mathcal{C}$

$$P \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$$

(= which represents a many sorted conjunctive LOGIC with equality)



Doctrine

a functor

$$P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$

from a *finite product* category $\mathscr{C}$ (**doctrine base**)

to the category of inf-semilattices and homomorphisms

is called **doctrine**.

Elementary Doctrine

a functor

$$P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$

from a *finite product* category $\mathscr{C}$ (**doctrine base**)

to the category of inf-semilattices and homomorphisms

s.t. for every object A in $\mathscr{C}$,

there is an **EQUALITY** object δ_A in $P(A \times A)$ (interpreting $x =_A y$)

such that for any predicate α in $P(X \times A)$

$$\exists\!\!\!\!\exists_{id_X \times \Delta_A}(\alpha) := P_{id_X \times \mathrm{pr}_1}(\alpha) \wedge_{A \times A} P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_A)$$

is a left adjoint:

$$\exists\!\!\!\!\exists_{id_X \times \Delta_A} \dashv P_{id_X \times \Delta_A}$$

ee-doctrine =Existential Elementary Doctrine

an **ee-doctrine** $P \colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

is an **Elementary doctrine** with **Existential Quantifiers**

$$\exists_{\mathrm{pr}} \colon P(A_1 \times A_2) \to P(A_i)$$

i.e. Left Adjoints to $P_{\mathrm{pr}_i} \colon P(A_i) \to P(A_1 \times A_2)$

for projections $\mathrm{pr}_i \colon A_1 \times A_2 \to A_i$ for $i = 1, 2$

+ **Beck-Chevalley** conditions, **Frobenius reciprocity**

P-partial equivalence relation on A object of $\mathscr{C}$

given an elementary doctrine $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSL}$

an object ρ of $P(A \times A)$ is a P-partial equivalence relation

iff it satisfies:

symmetry: $\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\rho)$

transitivity: $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\rho) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\rho)$

Exact completion of an elementary existential doctrine

from [Pitts'02,Maietti-Rosolini'15]:

for any **ee-doctrine** (= **elementary existential doctrine**)

$P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

we define its **exact/ee completion** as $\mathscr{T}_P$

which extends to a bi-adjunction
$$\begin{tikzcd} EED \arrow[rr, shift left=2, "\mathcal{T}_-"] \arrow[rr, phantom, "\perp"] & & \mathsf{Exact} \arrow[ll, shift left=2, "I"] \end{tikzcd}$$

EED= 2-category of **ee-doctrines**

Exact= 2-category of **exact categories**

ex/ee completion= exact completion of a ee-doctrine

For any **elementary existential** doctrine $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$
objects of $\mathscr{T}_P$ are pairs (A, ρ) with A object of $\mathscr{C}$ and ρ any P-**partial equivalence relation** in $P(A \times A)$
an arrow in $\mathscr{T}_P$ $\phi\colon (A, \rho) \to (B, \sigma)$ is a P-RELATION $\phi \in ObP(A \times B)$ which is a **Functional Relation** from reflexive elements in A to reflexive elements B

$\mathscr{T}_P$ of P *elementary existential doctrine*

For any elementary existential doctrine $P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

a **Functional Relation** from (A,ρ) to (B,σ) is a P-relation ϕ

preserving the equivalence relations

1) $\phi \leq P_{\langle p_1,p_1\rangle}(\rho) \wedge P_{\langle p_2,p_2\rangle}(\sigma)$

2) $P_{\langle p_1,p_2\rangle}(\rho) \wedge P_{\langle p_2,p_3\rangle}(\phi) \leq P_{\langle p_1,p_3\rangle}(\phi)$ in $P(A \times A \times B)$

3) $P_{\langle p_1,p_2\rangle}(\phi) \wedge P_{\langle p_2,p_3\rangle}(\sigma) \leq P_{\langle p_1,p_3\rangle}(\phi)$ in $P(A \times B \times B)$

4) $P_{\langle p_1,p_2\rangle}(\phi) \wedge P_{\langle p_1,p_3\rangle}(\phi) \leq P_{\langle p_2,p_3\rangle}(\sigma)$ in $P(A \times B \times B)$

5) $P_{\langle p_1,p_1\rangle}(\rho) \leq \exists_{p_2}(\phi)$ in $P(A)$

Notion of **tripos**

A **tripos** is an elementary Lawvere's doctrine

which is a first-order intuitionistic hyperdoctrine

+ weak power-objects

Notion of **tripos**

A **tripos** is an elementary Lawvere's doctrine

$P\colon \mathcal{C}^{op} \longrightarrow$ InfSL

which is a first-order intuitionistic hyperdoctrine

+

for every A object in $\mathbf{Ob}\mathcal{C}$

there exists a *weak* power-object $\mathbb{P}A \in \mathbf{Ob}\mathcal{C}$

a membership relation ε_A as an object of $P(A \times \mathbb{P}A)$

such that for every P- predicate ψ in $P(A \times Y)$

there is $\{\psi\}\colon Y \to \mathbb{P}A$ such that $P(id_A \times \{\psi\})(\varepsilon_A) = \psi$

the tripos-to-topos construction

Theorem:

if the doctrine P is a **tripos**

the ex/ee completion $\mathscr{T}_P$ of P

is an elementary topos.

main example:

Hyland's Effective topos **Eff**

with

$P_{\mathbf{Eff}} \colon Set^{op} \to$ **HA**

where $\mathbf{P}_{\mathbf{Eff}}(\mathbf{1}) \equiv$ Kleene first algebra

and $\mathbf{1}$ terminal object

Unifying exact completions via the ex/ ee completion

Theorem (**Ex/reg completion**= **exact completion** of a **ee-doctrine**)

the **exact completion** $\mathcal{C}_{ex/reg}$ of a **regular** category $\mathcal{C}$

is an instance of

the **exee completion**

of the "**subobjects doctrine**" of its base category $\mathcal{C}$

$$\boxed{\mathcal{C}_{ex/reg} \simeq \mathcal{T}_{\mathbf{Sub}_{\mathcal{C}}}}$$

Comprehension of a P-predicate α

A comprehension of a P*-predicate* α on A in $\mathscr{C}$

(i.e. of an object α of $P(A)$)

is an arrow $\{\!|\alpha|\!\} \colon \{x \in A \mid \alpha\} \to A$ in $\mathscr{C}$

satisfying $\top_X \leq P_{\{\!|\alpha|\!\}}(\alpha)$

(expressing that if $a \varepsilon \{x \in A \mid \alpha\} \quad \Rightarrow \quad \alpha(a)$)

+ a universal property:

for any $g \colon Y \to A$ s.t

$$\top_Y \leq P_g(\alpha)$$

there is a unique $h \colon Y \to \{x \in A \mid \alpha\}$ s.t.

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[dr, "\exists! h"'] & & A \\
& \{x \in A \mid \alpha\} \arrow[ur, "\{\!|\alpha|\!\}"'] &
\end{tikzcd}$$

full (weak) comprehension of a P-predicate α

A (weak) comprehension of a P*-predicate* α on A in $\mathscr{C}$

is **full**

when for α and β objects in $P(A)$

then $\alpha \leq \beta$ iff $\{\!|\alpha|\!\} \leq \{\!|\beta|\!\}$ in $wSub(A)$

full comprehensive completion of an doctrine

for any **doctrine**

$P: \mathcal{C}^{OP} \longrightarrow$ InfSL

we define its **full comprehensive completion** $P_c : Gr(P)^{\mathrm{op}} \longrightarrow$ InfSL

(with $Gr(P)$ Grothendieck completion of P)

which extends to a bi-adjunction
$$\begin{tikzcd} Doc \arrow[r, shift left=2, "(-)_c"] \arrow[r, phantom, "\perp"] & \mathsf{CD} \arrow[l, shift left=2, "I"] \end{tikzcd}$$

Doc= 2-category of **e-doctrines**

CD= 2-category of **full comprehensive doctrines**

free full comprehensive doctrine P_c of a doctrine P

For any *ee-doctrine*

$P\colon \mathscr{C}^{\text{op}} \longrightarrow \text{InfSL}$

the free **full comprehensive ee-doctrine** is

$P_c\colon Gr(P)^{\text{op}} \longrightarrow \text{InfSL}$

on the Grothendieck completion of P

objects of $P_c(\,(A,\alpha)\,)$	are objects γ in $P(A)$ with $\gamma \leq \alpha$.
on morphisms: $(P_c)_f(\gamma) \equiv P_f(\gamma) \wedge \beta$ for $f\colon (B,\beta) \rightarrow (A,\alpha)$	

Matching morphism equality with fibre equality

an elementary DOCTRINE

$$P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSL}$$

has comprehensive diagonals

iff

the diagonal arrow $\Delta_A\colon A \to A \times A$ is a comprehension of its equality δ_A

iff

morphism equality in $\mathscr{C}$ = *provable fibre equality* via *fibre equality* δ_B

$f =_{\mathscr{C}} g$ iff $\top_A \leq_A P_{\langle f,g\rangle}(\delta_B)$

free comprehensive diagonal doctrine P_x of a e-doctrine P

For any *e-doctrine* (= **elementary doctrine**)

$P \colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

the free **comprehensive diagonal doctrine** is

$P_x \colon {\chi_P}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

on the category χ_P

objects of χ_P = **objects** of $\mathscr{C}$

morphisms of χ_P =equivalence classes $[f]$ of morphisms f of $\mathscr{C}$ up to fibre equality

i.e. $\quad [f] = [f'] \quad$ iff $\quad \delta_A \leq P_{f \times f'}(\delta_B)$

which extends to a bi-adjunction
$$\begin{tikzcd} ED \arrow[r, shift left=2, "(-)_x"] \arrow[r, phantom, "\perp"] & \mathsf{EDD} \arrow[l, shift left=2, "I"] \end{tikzcd}$$

ED= 2-category of **e-doctrines**

EDD= 2-category of **e-doctrines** with **comprehensive diagonals**

Exact completion of a eecd-doctrine

from [Maietti-Rosolini'15]:

for any eecd-doctrine $P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

(i.e. any. *ee doctrine* with *(weak) full comprehensions* + comprehensive diagonals)

the category $\mathscr{E}_P$ is called the **ex**/eecd completion of P

since it gives rise to a bi-adjunction
$$\begin{tikzcd} EECDD \arrow[r, shift left=2, "\mathscr{E}_{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{Exact} \arrow[l, shift left=2, "I"] \end{tikzcd}$$

EECDD= 2-category of **eecd-doctrines**

Exact= 2-category of **exact categories**

exact completion $\mathscr{E}_P$ of a eecd-doctrine

For any **eecd-doctrine**

$P \colon \mathscr{C}^{\mathrm{op}} \longrightarrow$ InfSL

(i.e. any *ee doctrine* with *(weak) full comprehensions* + comprehensive diagonals)

objects of $\mathscr{E}_P$ are pairs (A, ρ) with A object of $\mathscr{C}$ ρ any P-equivalence relation in $P(A \times A)$
an arrow in $\mathscr{E}_P$ $\phi\colon (A,\rho) \to (B,\sigma)$ is a P-RELATION $\phi \in ObP(A \times B)$ which is a FUNCTIONAL RELATION from A to B $\top_A \leq \exists_{p_2}(\phi)$ preserving the equivalence relations

P-equivalence relation on A object of $\mathscr{C}$

given an elementary doctrine $P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

an object ρ of $P(A \times A)$ is a P-equivalence relation

iff it satisfies:

reflexivity: $\delta_A \leq \rho$

symmetry: $\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1 \rangle}(\rho)$

transitivity: $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(\rho) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\rho) \leq P_{\langle \mathrm{pr}_1, \mathrm{pr}_3 \rangle}(\rho)$

the Ex/eecd completion applies to eec-doctrines

Prop:

for any **ee-doctrine** P

with (weak) full comprehensions

the category $\mathscr{E}_P$ is exact

Key lemma 1

Theorem:

for any **ee-doctrine** P

with (weak) full comprehensions

the **ex/ee** completion is equivalent to the **ex/eecd** completion:

$\mathscr{T}_P$ is EQUIVALENT to $\mathscr{E}_P$

$$(A, \rho) \longrightarrow (X, P_{\eta\times\eta}(\rho))$$

with $\eta \equiv \{\!|P_{\langle id,id\rangle}(\rho)|\!\}: X \to A$

Key lemma 2

Theorem:

for any **ee-doctrine** $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{InfSL}$

the **ex**/**ee** completion is equivalent to the **ex**/**eecd** completion

of its free comprehensive doctrine :

$$\mathscr{T}_P \text{ is EQUIVALENT to } \mathscr{E}_{(P_c)_x} = \mathscr{E}_{(P_c)}$$

$$(A, \rho) \longrightarrow (\,(A, P_{\langle id, id\rangle}(\rho))\,,\rho)$$

Corollary :Ex/reg completion as an ex/ee completion

exact completion $\mathcal{C}_{ex/reg}$ of a **regular** category $\mathcal{C}$

is an instance of

exact completion

of the "subobjects doctrine" of its base category $\mathcal{C}$

since $\mathcal{C}_{ex/reg} \simeq \mathcal{E}_{\mathbf{Sub}_{\mathcal{C}}} \simeq \mathcal{T}_{\mathbf{Sub}_{\mathcal{C}}}$

Unifying exact completions via the ex/ee completion

Theorem (**Ex/wlex** completion= **ex/ee** completion of a ee-doctrine)

the **exact completion** $\mathcal{C}_{ex/wlex}$ of a weakly left exact **finite product** category $\mathcal{C}$

is an instance of

the **ex/ee** completion wSub of the "**weak subobject doctrine**" $wSub_{\mathcal{C}}$ of $\mathcal{C}$

(i.e. $\mathbf{wSub}_{\mathcal{C}}(A)$ =**poset reflection** of $\mathcal{C}/A$

$\mathbf{\Psi}_{\mathcal{C}}(f)$ = a(ny) **weak pullback** of an arrow in $\mathcal{C}/A$

$$\boxed{\mathcal{C}_{ex/wlex} \simeq \mathcal{T}_{\mathbf{wSub}_{\mathcal{C}}}}$$

Ex/wlex completion as an elementary quotient completion

from [M.Rosolini13]

Theorem

For any weakly left exact **finite product** category $\mathcal{C}$

the **exact completion** $\mathcal{C}_{ex/wlex}$ of a weakly left exact **finite product** category $\mathcal{C}$

is equivalent to the base category $\mathcal{Q}_{\mathbf{wSub}_{\mathcal{C}}}$

of the **elementary quotient completion** of the "**weak subobject doctrine**" $wSub_{\mathcal{C}}$ of $\mathcal{C}$

$$\boxed{\mathcal{C}_{ex/wlex} \simeq \mathcal{Q}_{\mathbf{wSub}_{\mathcal{C}}}}$$

Elementary Quotient Completion

for any **elementary doctrine** $P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$
its **Elementary Quotient Completion** is the elementary doctrine $$\overline{P}\colon {\mathscr{Q}_P}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$ which *freely* extends P with *stable* effective quotients

Base of the Elementary Quotient Completion $\mathcal{Q}_P$

objects of $\mathcal{Q}_P$ are quotient presentations/setoids

(A, ρ) with ρ a P-equivalence relation on A

(written in the logic of P)

arrows of $\mathcal{Q}_P$ are equivalence classes $\lfloor f \rceil : (A, \rho) \to (B, \sigma)$

of arrows $f\colon A \to B$ in $\mathscr{C}$ *preserving the equivalence relations*

$$\rho \leq_{A \times A} P_{f \times f}(\sigma)$$

such that

$$f =_{\mathcal{Q}_P} g \qquad \text{iff} \qquad \rho \leq P_{f \times g}(\sigma)$$

Fibres of lifted doctrine of the elementary quotient completion

for (A, ρ) object of $\mathcal{Q}_P$ the fibres of the lifted doctrine on the elementary quotient completion

$$\overline{P}(A, \rho) := \mathcal{D}es_\rho$$

are descent data $\alpha \in \mathcal{D}es_\rho$

i.e. P-predicates preserving the equivalence relation ρ

$$P_{\mathrm{pr}_1}(\alpha) \wedge \rho \leq P_{\mathrm{pr}_2}(\alpha)$$

with projections $\mathrm{pr}_1, \mathrm{pr}_2 \colon A \times A \to A$.

Embedding P into its elementary quotient completion $\overline{P}$

There is a 1-arrow embedding $(J, j)\colon P \to \overline{P}$ between elementary doctrines

$$\begin{array}{|llcr|} J\colon & \mathscr{C} & \to & \mathscr{Q}_P \\ & A & \longrightarrow & (A, \delta_A) \\ f\colon A \to B & & \mapsto & f\colon (A, \delta_A) \to (B, \delta_B) \end{array}$$

but to make this embedding faithful

we need to ask that P has comprehensive diagonals

i.e. the diagonal arrow $\Delta_A\colon A \to A \times A$ is a comprehension of its equality δ_A

Universal Property of Elementary Quotient Completion

For any elementary doctrine with comprehensive diagonals $P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

pre-composition with the 1-arrow

$$\begin{tikzcd}
\mathscr{C}^{\mathsf{op}} \arrow[d, "J"'] \arrow[rrd, "P"] \arrow[rd, phantom, "j \downarrow \cdot"] & & \\
{\mathscr{Q}_P}^{\mathsf{op}} \arrow[rr, "\overline{P}"'] & & \mathsf{InfSL}
\end{tikzcd}$$

in EqD induces an essential equivalence of categories

$$- \circ (J, j)\colon \mathsf{QD}(\overline{P}, X) \equiv \mathsf{EqD}(P, X)$$

for every X in QD.

P is in EqD i.e. is elementary + comprehensive diagonals

$\overline{P}$ is in QD i.e. is in EqD + stable effective quotients

Quotient relative to a doctrine

given an elementary doctrine $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$

a quotient of the P-equivalence relation ρ on A

is a $\mathscr{C}$-arrow $\qquad q\colon A \to A/\rho$

s.t.

$$\rho \leq P_{q\times q}(\delta_C)$$

+

for $g\colon A \to Z$ $\quad$ s.t. $\quad \rho \leq P_{g\times g}(\delta_Z)$

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "q"'] & & Z \\ & A/\rho \arrow[ur, "\exists! h"'] & \end{tikzcd}$$

Effective quotients

A quotient $q\colon A \to A/\rho$

of the P-equivalence relation ρ

is effective

iff

ρ is its P-kernel

i.e. $\rho = P_{q \times q}(\delta_B)$

Elementary Quotient Completion NOT exact

$\boxed{\begin{array}{c} \mathcal{Q}_P \text{ is not always EXACT whilst REGULAR!!} \\ \text{(every } Sub_{\mathcal{Q}_P}\text{-equivalence relation in } \mathcal{Q}_P \\ \text{has a stable coequalizer but NOT effective)} \end{array}}$

for elementary existential doctrine P with (*weak*) full comprehensions

Elementary Quotient Completion NOT exact

EXAMPLES of NOT exact $\mathcal{Q}_P$ with:

P=Strong subobjects on partitioned assemblies

$\Rightarrow$ the base category $\mathcal{Q}_P$ is the full subcategory of assemblies

in **Hyland's Effective Topos** (see [M.Pasquali-Rosolini19])

motivating examples 

P= logic of Coquand's Calculus of Constructions

P= logic of the intensional level of the **Minimalist Foundation**

in [M09]

$\Rightarrow$ in both examples the base category $\mathcal{Q}_P$ is the so called setoid model

When is the Elementary Quotient Completion $\mathcal{Q}_P$ exact?

from [Maietti-Rosolini'16]

$\mathcal{Q}_P$ exact

$\Updownarrow$

when any monomorphism in the regular category $\mathcal{Q}_P$ is a comprehension

$\Updownarrow$

$\overline{P}\colon \mathcal{Q}_P{}^{\mathsf{op}} \longrightarrow$ InfSL is equivalent to $Sub\colon \mathcal{Q}_P{}^{\mathsf{op}} \longrightarrow$ InfSL

$\Updownarrow$

the rule of choice holds in the internal language of P

$\Updownarrow$

the rule of UNIQUE choice holds in the internal language of quotient doctrine $\overline{P}$

Rule of choice

in a theory $\mathbf{T}$

if

$$\exists y \in B \; R(x, y) \; [x \in \Gamma]$$

is true in $\mathbf{T}$

$$\Downarrow$$

there exists a function term

$$f(x) \in B \; [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x)) \; [x \in \Gamma]$$

is true in $\mathbf{T}$.

Rule of unique choice

in a theory $\mathbf{T}$

if $\exists! y \in B \; R(x,y) \; [x \in \Gamma]$

is true in $\mathbf{T}$

$$\Downarrow$$

there exists a function term

$$f(x) \in B \; [x \in \Gamma]$$

in $\mathbf{T}$ such that

$$R(x, f(x)) \; [x \in \Gamma]$$

is true in $\mathbf{T}$.

why $\mathscr{Q}_P$ is not exact in essence

existential quantification in

$$\overline{P}\colon {\mathscr{Q}_P}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$

$$\neq$$

regular existential quantification in the subobject doctrine

$$Sub\colon {\mathscr{Q}_P}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$$

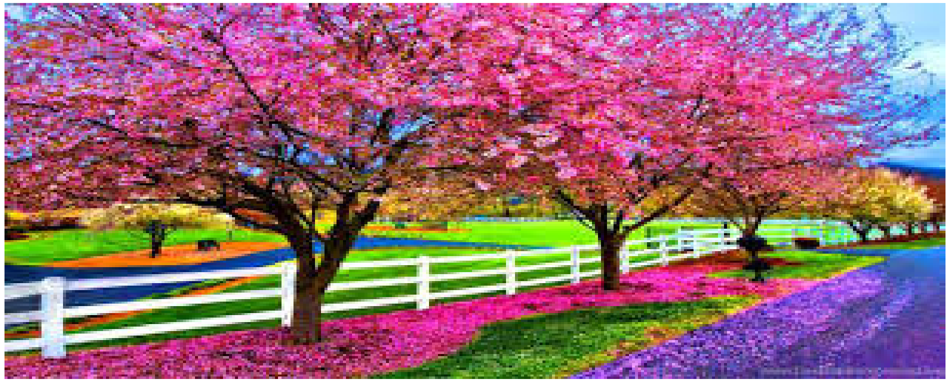
a htripos-to-quasi-topos construction

j.w.w. **F. Pasquali** and **G. Rosolini**

for any tripos $P: \mathcal{C}^{op} \longrightarrow$ InfSL	+ weak full comprehensions + $\mathcal{C}$ is slicewise weakly cartesian closed (= weakly closed w.r.t weak products in the slice cats) + finite distributive coproducts in $\mathcal{C}$ + a natural number object in $\mathcal{C}$ called **htripos**
the **elementary quotient completion** $\mathcal{Q}_P$ of P is an **arithmetic quasitopos**.	

Main examples of htripos-to-quasi-topos construction

the setoid category over the calculus behind the proof-assistant **Coq**
the category of *assemblies* in Hyland's Effective Topos
the category of Scott's equilogical spaces



toposes as *htripos-to-quasi-topos constructions*

j.w.w. **F. Pasquali** and **G. Rosolini**

Theorem:

for any *hyper-tripos* $P\colon \mathcal{C}^{op} \longrightarrow$ InfSL

---

the elementary quotient completion $\mathcal{Q}_P$ of P

is a topos (and not only a quasi-topos)

iff

$\mathcal{Q}_P$ is equivalent to the exact completion $\mathcal{C}_{ex/lex}$ of $\mathcal{C}$

iff

P satisfies the **rule of choice**

toposes as ex/lex completions

Theorem (**j.w.w. Davide Trotta**)

A tripos-to-topos construction τ_P is a ex/lex completion

iff τ_P is equivalent to $\tau_{P'}$

with $P' \colon \mathcal{C}^{op} \longrightarrow$ InfSL a generalized existential completion

as defined in [Trotta20]

with respect the class of morphisms of a lex subcategory of $\mathcal{C}$

Regular completion of a eecd-doctrine

from **[Maietti-Pasquali-Rosolini'17]**:

for any **eecd-doctrine**

(= **elementary existential doctrines** with **full comprehensions** + **comprehensive diagonals**)

$P : \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$

we define its **reg/eecd** completion as $\mathscr{E}\mathscr{F}(P)$

which extends to a bi-adjunction
$$\begin{tikzcd} EECD \arrow[r, shift left=2, "\mathscr{E}\mathscr{F}(-)"] \arrow[r, phantom, "\perp"] & \mathsf{Reg} \arrow[l, shift left=2, "I"] \end{tikzcd}$$

EECD= 2-category of **eecd-doctrines**

Reg= 2-category of **regular categories**

regular completion $\mathscr{E}\mathscr{F}P$ of an eecd-doctrine

For any **eecd-doctrine**

$P: \mathscr{C}^{\mathrm{op}} \longrightarrow$ InfSL

(i.e. any. *ee doctrine* with *(weak) full comprehensions* + comprehensive diagonals)

objects of $\mathscr{E}\mathscr{F}(P)$	are objects of $\mathscr{C}$
an arrow in $\mathscr{E}_P$	ϕ: $A \to B$ is a P-RELATION

$$\phi \in ObP(A \times B)$$

which is a FUNCTIONAL RELATION from A to B

Regular completion of a eed-doctrine

from **[Maietti-Pasquali-Rosolini'17]**:

for any **eed-doctrine** (= **elementary existential doctrines** with comprehensive diagonals)

$P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSL}$

we define its **reg/eed** completion as $\mathscr{E}\mathscr{F}(P_c)$

which extends to a bi-adjunction $$\begin{tikzcd} EED \arrow[r, shift left=2, "\mathscr{E}\mathscr{F}_{(-)_c}"] \arrow[r, phantom, "\perp"] & \mathsf{Reg} \arrow[l, shift left=2, "I"] \end{tikzcd}$$

EED= 2-category of **eed-doctrines**

Reg= 2-category of **regular categories**

the Regular completion applies to any elementary existential doctrine

from [Maietti-Psquali-Rosolini'17]:

for any **ee-doctrine** (= **elementary existential doctrine**)
$P\colon \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{InfSL}$
we define its **reg/ee** construction as $\mathscr{E}\mathscr{F}(P_c)$
since $\mathscr{E}\mathscr{F}(P_c) = \mathscr{E}\mathscr{F}((P_x)_c)$

Decomposition of exact completion behind the tripos-to-topos construction

$$
\begin{tikzcd}
EED \arrow[rr, shift right, "\mathcal{T}_{-}"'] \arrow[rr, phantom, "\top"] \arrow[d, shift right, "\text{+ full comprehension}"'] \arrow[d, phantom, "\dashv"] & & \mathsf{Exact} \arrow[ll, shift right] \arrow[dd, shift right] \arrow[dd, phantom, "\vdash"] \\
EECD \arrow[u, shift right] \arrow[d, shift right, "\text{+ comprehensive diagonals}"'] \arrow[d, phantom, "\dashv"] & Reg \arrow[ur, "\text{ex/reg completion}"] & \\
EECDD \arrow[u, shift right] \arrow[ur, "\text{regular completion}"'] \arrow[rr, shift right, "\text{elementary quotient completion}"'] \arrow[rr, phantom, "\top"] & & EECDQ \arrow[ll, shift right] \arrow[uu, shift right, "\text{regular completion}"']
\end{tikzcd}
$$

various notions of **predicative topos***-like structures*

As usual in predicative mathematics

for a **predicative versions** of *classical impredicative concepts*

different proposals of **predicative topos/quasi-topos**

may co-exist.

Toposes versus Quasi-Toposes

predicative generalization of **topos**

are needed to build realizability models

to guarantee constructivity for **intuitionistic** predicative mathematics

Toposes versus Quasi-Toposes

No existing predicative generalization of a **topos**

can be considered a **foundation** for **classical** predicative mathematics

because

Prop. **Boolean** predicative toposes are **toposes**

with respect to the known notions in the literature

including ours

ONLY impredicative boolean predicative topos exist

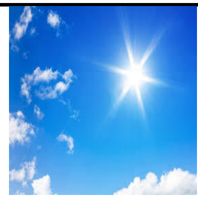
The given notions of **predicative topos**

are NOT **adequate foundations** for

classical predicative mathematics a' la Weyl-Feferman

$\Longrightarrow$ we need to predicatively generalize the notion of quasi-topos!

Our criteria to define a **predicative topos**

a *predicative generalization* of the notion of **topos** 

should satisfy

its generic internal language TT_{ptop} is predicative à la Feferman
its generic internal language TT_{ptop} is equivalent to that of **toposes** when a resizing rule/reducibility axiom is added to it
as examples– it includes categorical structures of sets/classes of relevant **predicative foundations** of mathematics
it allows a *straightforward generalizatio*n to a notion of predicative quasi-topos.

key point *in our notion of* **predicative elementary topos/quasi-topos**

to avoid *impredicativity*

we declare

the power-object $\mathcal{P}(A)$ of a **set** A is a **collection** (or class)

ENTITIES in our predicative topos-like structures

$$\begin{tikzcd} \text{small propositions} \arrow[d, hook] \\ \text{propositions} \end{tikzcd} \qquad\qquad \begin{tikzcd} \text{sets} \arrow[d, hook] \\ \text{collections} \end{tikzcd}$$

where

small propositions = propositions with restricted quantifiers

as in

"*Algebraic Set Theory*" *A. Joyal - I. Moerdijk*, OUP, 1995

Some notations on fibered categories

By the word fibration we mean a fibred category

$$\mathcal{S} \xrightarrow[p]{} \mathcal{C}$$

such that for any object $\mathcal{S}$-object B and any $\mathcal{C}$-morphism

$$f\colon Y \to p(B)$$

there exists a cartesian arrow $u\colon A \to B$ over f.

We use the notation

$$cod\colon \mathcal{C}^{\to} \to \mathcal{C}$$

to denote the codomain fibration of a finite limits category $\mathcal{C}$.

Our Predicative Generalization of Elementary topos

A **predicatively generalized elementary topos** - for short **predicative topos** is given by a fibration

$$\pi_{\mathcal{S}} : \mathcal{S} \to \mathcal{C}$$

satisfying the following properties: (the categorical semantics of TT_{ptop})

- the category $\mathcal{C}$ has finite limits

  ($\mathcal{C}$ is meant to be the category of collections)

- the subobject doctrine $\mathbf{Sub}_{\mathcal{C}}$ associated to $\mathcal{C}$ is a first order Lawvere hyperdoctrine

  (represents the logic over collections)

- the fibration $\pi_{\mathcal{S}} : \mathcal{S} \to \mathcal{C}$ is a FULL sub-fibration of the codomain fibration on $\mathcal{C}$

  ($\pi_{\mathcal{S}}$ represent family of sets indexed over collections)

$$
\begin{tikzcd}
\mathcal{S} \arrow[rr, hook, "i"'] \arrow[dr, "\pi_{\mathcal{S}}"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\
& \mathcal{C} &
\end{tikzcd}
$$

i.e. i is an inclusion functor preserving cartesian morphisms and making the diagram commute.

- for each object A of $\mathcal{C}$ the fibre $\mathcal{S}_A$ of $\pi_{\mathcal{S}}$ over A is a locally cartesian closed pretopos;

- for any morphism $f : A \to B$ the substitution functor

$$f^* : \mathcal{S}_B \to \mathcal{S}_A$$

preserves the LCC pretopos structure;

- for each object A of $\mathcal{C}$ the embedding of each fibre $\mathcal{S}_A$ into $\mathcal{C}/A$ preserves the LCC pretopos structure;

- there is a $\mathcal{C}$-object Ω
  classifying the subobjects of $\mathcal{C}$ which are in $\mathcal{S}$:

  $$\mathsf{Sub}_{\mathcal{S}} \simeq \mathcal{C}(-, \Omega)$$

  where $\mathsf{Sub}_{\mathcal{S}}(A)$ is the full subcategory of $\mathsf{Sub}_{\mathcal{C}}(A)$ of those subobjects which are represented by objects in $\mathcal{S}$;

- for every $\mathcal{C}$-object A,
  for every object α: $X \to A$ in $\mathcal{S}$,
  there is an *exponential object* $(\pi_\Omega)^\alpha$ in $\mathcal{C}/A$
  where π_Ω: $A \times \Omega \to A$ is the first projection, i.e. there is a natural isomorphism

  $$\mathcal{C}/A(- \times \alpha\,,\ \pi_\Omega\,) \simeq \mathcal{C}/A(-\,,\ (\pi_\Omega)^\alpha\,)$$

  as functors on $\mathcal{C}/A$.

Our examples of predicatively generalized elementary toposes

In our next examples of predicatively generalized elementary toposes

$\pi_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{C}$

we just specify $\mathcal{C}$ and $\mathcal{S}$

since $\pi_{\mathcal{S}}$ must be the *restriction* of the codomain fibration

Elementary toposes are examples of our structures

An **elementary topos** $\mathcal{T}$ is a **predicatively generalized elementary topos**

with collections=sets:

$\mathcal{S} = \mathcal{C}^{\rightarrow} = \mathcal{T}^{\rightarrow}$

$\pi_{\mathcal{S}} = cod_{\mathcal{T}}$

i.e.

$\pi_{\mathcal{T}} : \mathcal{T}^{\rightarrow} \rightarrow \mathcal{T}$

To build examples of predicative toposes

we need of a predicative analogue

of Johnstone-Hyland-Pitts tripos-to-topos construction

we adopt

the **exact on lex completion**

viewed as an elementary quotient completion

to better compute with it.

An example with Feferman's Theory of NON-iterative fixpoints

In **Feferman's Theory of NON-iterative fixpoints** $\widehat{ID_1}$

we use *formulas defining fixpoints* of so called admissible formulas

to define

a **universe of $I\hat{D}_1$-sets**	$\mathsf{U_0}^{I\hat{D}_1}$
a notion of $I\hat{D}_1$**-small proposition** as a $I\hat{D}_1$-set which is at most singleton	

exactly as that used in

[I. Ishihara, M.E.M., S. Maschio, T.Streicher'18]

"Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice", *AML*.

A Feferman's predicative version of Hyland's Effective Topos

from

M.E. Maietti and S. Maschio "A predicative variant of Hyland's Effective Topos" to appear in JSL 2021

Let $\mathbf{Rec}^{I\hat{D}_1}$ be the following category

objects	definable classes in $I\hat{D}_1$ (i.e. subclasses of natural numbers defined by formulas $\phi(x)$ up to renaming of variables)
morphisms	recursive functions in $\widehat{ID_1}$ denoted by numerals
morphism equality	extensional equality

we define a predicatively generalize elementary topos

meant to be a predicative version of *Hyland*'s Effective Topos

with:

$\mathcal{C}_{pEff}^{\widehat{ID_1}}$	= the exact on lex completion $\mathbf{Rec}^{I\hat{D}_1}$ viewed as elementary quotient completion
Objects of $\mathcal{S}_{pEff}$	= objects of $\mathcal{C}_{pEff}^{\rightarrow}$ isomorphic in the fibre over their codomain $A_=$ to projections of $A_=$-indexed families of objects in $\mathcal{C}_{pEff}$ whose support is in $\mathsf{U_0}^{I\hat{D}_1}$ and whose equivalence relation is a $I\hat{D}_1$-small proposition

the base of the **predicative Effective topos** *à la* **Feferman**

$$\mathbf{pEff} \equiv \mathcal{Q}_{wSub_{\mathbf{Rec}^{\mathbf{I\hat{D}_1}}}} = (\mathbf{Rec}^{I\hat{D}_1})_{ex/lex}$$

Kleene realizability interpretation in **Eff** *and in* **pEff**

the interpretation of the *logical connectives and quantifiers*

in the hyperdoctrine structure of the subobject functor

is equivalent to Kleene realizability interpretation of intuitionistic logic.

a *categorically motivation for this* is in

M. E.M, F. Pasquali, G. Rosolini:

Elementary Quotient Completions, Church's Thesis and Partitioned Assemblies.

Log. Methods Comput. Sci. 15(2) (2019)

Relating **pEff** *to* **Eff**

the category of collections of our predicatively generalized elementary topos in $\widehat{ID_1}$

$$\mathbf{pEff} \equiv \mathcal{C}_{pEff}^{\widehat{ID_1}}$$

can be mapped (interpreted) in *Hyland*'s Effective Topos Eff

thanks to the fact that Eff is an exact on lex completion on partioned assemblies

by mapping (interpreting) the category $\mathbf{Rec}^{I\hat{D}_1}$ of recursive functions in $\widehat{ID_1}$

in the corresponding category of subsets of natural numbers and recursive functions in Eff.

A constructive generalized predicative version of Hyland's Effective Topos

j.w.w **Samuele Maschio**

Let $\mathbf{Rec}^{CZF+REA}$ be the following category

objects	definable classes in $CZF + REA$ (i.e. subclasses of natural numbers defined by formulas $\phi(x)$ up to renaming of variables)
morphisms	recursive functions in $\widehat{ID_1}$ denoted by numerals
morphism equality	extensional equality

the category $\mathbf{Rec}^{CZF+REA}$ supports a (non-categorical) interpretation of the intensional level of **MF**

in [M.Maschio-Rathjen21]: A realizability semantics for inductive formal topologies, Church's Thesis and Axiom of Choice, LOMECS (2021)

How to build a constructive predicative version of Hyland's Effective Topos

j.w.w **Samuele Maschio**

we can define a constructive generalized predicative elementary topos

meant to be a constructive generalized predicative version of *Hyland*'s Effective Topos

as $\mathcal{C}_{pEff}$ done on $\widehat{ID_1}$

guided by the interpretation in [M.Maschio-Rathjen21]

instead of that in [I. Ishihara, M.E.M., S. Maschio, T.Streicher'18]

the base of the **constructive predicative Effective topos**

$$\mathbf{cpEff} \equiv \mathcal{Q}_{wSub_{\mathbf{Rec}^{\mathbf{CZF+REA}}}} = (\mathbf{Rec}^{\mathbf{CZF+REA}})_{ex/lex}$$

Future work

- provide most general predicative tripos-to-topos construction including that used to build our predicative Effective Topos
- provide most general predicative tripos-to-quasi-topos construction including that used to build our predicative quasi-topos modelling the extensional level of the **Minimalist Foundation** in [M09]