Model structures from models of HoTT

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Models of HoTT from QMC

Quillen model categories are used to model homotopy theory. They can also be used to model Homotopy Type Theory.

- Hofmann-Streicher groupoid model can be viewed *post hoc* in terms of a QMS on Gpd. This also works for $\mathsf{Gpd}^{\mathbb{C}}$ (*stacks* as in Shulman's Topos Colloquium)
- A-Warren: models in general WFS and QMC (Id)
- Van den Berg-Garner: special WFS on sSet and Spaces (Π)
- Voevodsky: the Kan QMS on sSet (U)
- Shulman: every ∞-topos models HoTT

At each step, a more specialized QMC led to "better" models, with Id, Σ, Π and eventually univalent U.

Finally, Shulman showed just what was needed to build a model.

QMC from models of HoTT

One can also *reverse* the model construction, in a certain sense: start from a model of HoTT and construct from it a QMC.

We then have the following:

Conjecture

The resulting QMC presents an ∞-topos.

This reinforces the idea of HoTT as the *internal language* of ∞-topoi, just as IHOL is the internal language of 1-topoi.

$$\text{HoTT}/\infty\text{-topos} \quad :: \quad \text{IHOL}/\text{topos}$$

It also gives a strange new way of constructing a QMC, using ideas from type theory (today's main emphasis).

Models of HoTT

The models of HoTT that we start with are formulated in IHOL or *extensional* type theory. This is like a *translation* of one logical system into another. These models can even be formalized.

We can also describe this kind of model construction semantically in an elementary topos.

Definition (*pace* Orton-Pitts)

A *premodel of HoTT* in a topos $\mathcal{E}$ consists of $(\mathbb{I}, \Phi, \mathsf{V})$, where:

- $\mathbb{I}$ is an interval $1 \rightrightarrows \mathbb{I}$
  (that is *tiny* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ and ...)
- Φ is a representable class of monos $\Phi \hookrightarrow \Omega$
  (that is a *dominance* and ...)
- $\dot{\mathsf{V}} \to \mathsf{V}$ is a *universal small map*,
  (that is closed under Σ, Π and ...)

We will instead use this set-up to construct a QMS on $\mathcal{E}$.

QMS from a premodel

The construction of a QMS $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ from a premodel $(\mathbb{I}, \Phi, \mathsf{V})$ has so far been done for the following cases of a topos $\mathcal{E}$:

Cubical sets $\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$ for $\mathbb{C}$:

- Dedekind cubes (Sattler)
- Cartesian cubes (A)
- Cartesian cubes with equivariance (ACCRS)

Other examples in progress include:

- cubical realizability (AAFS)
- (higher) stacks (Coquand)
- simplicial and cubical (pre)sheaves
- quasi-categories

The premodel $(\mathbb{I}, \Phi, \mathsf{V})$ in $\mathcal{E}$

For today, the **topos** $\mathcal{E}$ will be the *Dedekind cubical sets*

$$\mathcal{E} = \mathsf{Set}^{\mathbb{C}^{op}}$$

where $\mathbb{C} \hookrightarrow \mathsf{Cat}$ is the full subcategory on the finite powers of $\mathbb{2}$.
This fp-category $\mathbb{C}$ is the *Lawvere theory* of distributive lattices.

The **interval** $\mathbb{I} = \mathsf{y}(\mathbb{2})$ therefore has *connections*,

$$\wedge, \vee : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}\,,$$

in addition to the *endpoints*,

$$\delta_0, \delta_1 : 1 \longrightarrow \mathbb{I}\,.$$

This is because $\mathbb{I} \times \mathbb{I} = \mathsf{y}(\mathbb{2}) \times \mathsf{y}(\mathbb{2}) \cong \mathsf{y}(\mathbb{2} \times \mathbb{2})$ and $1 \cong \mathsf{y}(\mathbb{1})$.

Moreover $\mathbb{I}$ is indeed *tiny*, since $\mathbb{C}$ is closed under finite products.

The premodel $(\mathbb{I}, \Phi, V)$ in $\mathcal{E}$

The subobject $\Phi \hookrightarrow \Omega$ is called the **cofibration classifier**. Logically, it determines a modal operator $\mathrm{cof} : \Omega \to \Omega$ on propositions. It will determine which monos are cofibrations in our model structure. The *dominance* law

$$\mathrm{cof}(p) \wedge (p \Rightarrow \mathrm{cof}(q)) \Rightarrow \mathrm{cof}(p \wedge q)$$

will imply that these monos are closed under composition.

For today, we will take $\Phi = \Omega$ to be simply *all* monos.

For some applications other choices are better. For example, one can take those monos $m : A \rightarrowtail B$ in $\mathrm{Set}^{\mathbb{C}^{op}}$ whose components $m_c : A(c) \rightarrowtail B(c)$ are *complemented subobjects* in Set. This condition is non-trivial if Set is not assumed to be boolean, as in realizability or constructive mathematics, or in a relative setting where e.g. $\mathrm{Set} = \mathrm{Sh}(X)$ (simplicial sheaves).

The premodel $(\mathbb{I}, \Phi, \mathsf{V})$ in $\mathcal{E}$

For $\dot{\mathsf{V}} \to \mathsf{V}$ we take a (Hofmann-Streicher) **universe** determined by a large enough cardinal κ. Thus let $\mathsf{Set}_\kappa \hookrightarrow \mathsf{Set}$ be the full subcategory of sets of size $< \kappa$, called *small*. Then let:

$$\mathsf{V}(c) = \{S : (\mathbb{C}/c)^{\mathrm{op}} \longrightarrow \mathsf{Set}_\kappa\},$$

the *set* of small presheaves on $\mathbb{C}/c$

$$\dot{\mathsf{V}}(c) = \{\dot{S} : (\mathbb{C}/c)^{\mathrm{op}} \longrightarrow \dot{\mathsf{Set}}_\kappa\},$$

the set of small *pointed* presheaves on $\mathbb{C}/c$.

Definition

An object A in $\mathcal{E}$ is *small* if its values $A(c)$ are small, for all $c \in \mathbb{C}$. A map $A \to X$ in $\mathcal{E}$ is *small* if its fibers $A_x = x^*A$ are small, for all $x : \mathsf{y}(c) \to X$.

The premodel $(\mathbb{I}, \Phi, \mathsf{V})$ in $\mathcal{E}$

Proposition

For every small map $A \to X$ there is a classifying map $a : X \to \mathsf{V}$ fitting into a pullback diagram of the form:

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{V}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \xrightarrow[a]{} & \mathsf{V} \end{array}$$

The universe V is closed under Σ and Π because small maps are closed under the adjoints $\Sigma \dashv * \dashv \Pi$ to pullback along small maps.

QMS on $\mathcal{E}$ from $(\mathbb{I}, \Phi, \mathsf{V})$

A Quillen model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is then constructed in 3 steps:

1. use Φ to determine a wfs $(\mathcal{C}, \mathsf{TFib})$,
2. use (1) and $\mathbb{I}$ to determine a wfs $(\mathsf{TCof}, \mathcal{F})$,
3. let $\mathcal{W} = \mathsf{TFib} \circ \mathsf{TCof}$ and show 3-for-2.

It is mainly the third step that involves new ideas from type theory.

QMS on $\mathcal{E}$ from $(\mathbb{I}, \Phi, \mathrm{V})$

Specifically, we use a *universal fibration* $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ as follows.

(i) construct $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ as $\dot{\mathrm{V}} \to \mathrm{V}$ equipped with a *type of fibration structures*,

(ii) show that $\dot{\mathrm{U}} \twoheadrightarrow \mathrm{U}$ is *univalent*,

(iii) univalence implies that U is fibrant,

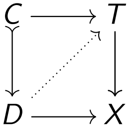
(iv) fibrant U implies 3-for-2 for $\mathcal{W}$.

The idea of getting a QMS from univalence is due to Sattler.

1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

The **cofibrations** $\mathcal{C}$ are just the monos $C \rightarrowtail D$ (for today).

The **trivial fibrations** TFib are the maps $T \to X$ with the *right lifting property* against the cofibrations.

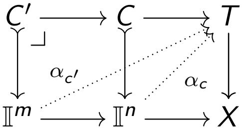


1. The cofibration wfs $(\mathcal{C}, \mathsf{TFib})$

Using the classifier Φ for $\mathcal{C}$, one can show:

Proposition

A map $T \to X$ is in TFib *iff it can be equipped with a coherent choice α of diagonal fillers for all cofibrations $c : C \rightarrowtail \mathbb{I}^n$ (and all filling problems).*



Such an α is then an *algebraic structure* on $T \to X$, and there is a *classifying type* $\mathsf{TFib}(T) \to X$ for such structures.

2. The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

The **fibrations** $\mathcal{F}$ are defined in terms of the trivial fibrations by

$$f \in \mathcal{F} \qquad \text{iff} \qquad (\delta \Rightarrow f) \in \mathsf{TFib}$$

where $\delta \Rightarrow (-)$ is the pullback-hom functor $\mathcal{E}^{\mathbb{2}} \to \mathcal{E}^{\mathbb{2}}$ given by an endpoint $\delta : 1 \to \mathbb{I}$.

The **trivial cofibrations** TCof are maps with the LLP against $\mathcal{F}$.

Small generators: For $c : C \rightarrowtail \mathbb{I}^n$ a cofibration and $\delta : 1 \to \mathbb{I}$ an endpoint, let

$$c \otimes \delta : [c] \rightarrowtail \mathbb{I}^n \times \mathbb{I}$$

be the *(partial) open box* with $[c] = \mathbb{I}^n +_C (C \times \mathbb{I})$.

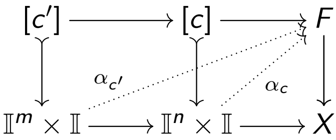
(*Think:* $\sqcup \rightarrowtail \boxdot$)

2. The fibration wfs $(\mathsf{TCof}, \mathcal{F})$

Again we have:

Proposition

A map $F \to X$ is in $\mathcal{F}$ iff it can be equipped with a coherent choice α of diagonal fillers for all open boxes $[c]$ (and all filling problems),



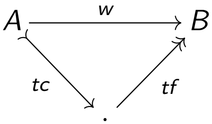
for all maps $\mathbb{I}^m \to \mathbb{I}^n$.

Such an α is again an *algebraic structure* on $F \to X$, and again there is a *classifying type* $\mathsf{Fib}(F) \to X$ for such structures.

3. The weak equivalences $\mathcal{W}$

Definition

A map $w : A \to B$ is a **weak equivalence** if it factors as a trivial cofibration followed by a trivial fibration.



It is then easy to show:

Proposition

$$\mathrm{TCof} = \mathcal{W} \cap \mathcal{C}$$
$$\mathrm{TFib} = \mathcal{W} \cap \mathcal{F}$$

Corollary

If $\mathcal{W}$ satisfies 3-for-2, then $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS on $\mathcal{E}$.

(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Definition

A *universal fibration* is a small fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ such that every small fibration $A \twoheadrightarrow X$ is a pullback of $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ along a *classifying* map $X \to \mathsf{U}$.

$$\begin{array}{ccc} A & \longrightarrow & \dot{\mathsf{U}} \\ \big\downarrow\!\!\!\downarrow & \lrcorner & \big\downarrow\!\!\!\downarrow \\ X & \longrightarrow & \mathsf{U} \end{array}$$

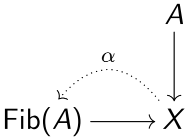
We will construct a universal fibration using the classifying type for fibration structures.

(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

For any $A \to X$ there is a *classifying type for fibration structures*,

$$\mathsf{Fib}(A) \longrightarrow X,$$

sections of which correspond to fibration structures α on $A \to X$.



NB: $\mathsf{Fib}(A) \to X$ is small when $A \to X$ is small.

(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

The map $\mathsf{Fib}(A) \to X$ is stable under pullback,

$$f^*\mathsf{Fib}(A) \cong \mathsf{Fib}(f^*A).$$

Thus the bottom square below is also a pullback.

$$\begin{array}{ccc}
f^*A & \longrightarrow & A \\
\downarrow & & \downarrow \\
Y & \xrightarrow[f]{} & X \\
\uparrow & & \uparrow \\
\mathsf{Fib}(f^*A) & \longrightarrow & \mathsf{Fib}(A)
\end{array}$$

The construction of $\mathsf{Fib}(A)$ uses the *root functor* $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$.

(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Now let U be the type of fibration structures on $\dot{\mathsf{V}} \to \mathsf{V}$,

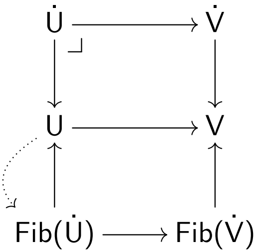
$$\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}}) \longrightarrow \mathsf{V}.$$

Then define $\dot{\mathsf{U}} \to \mathsf{U}$ by pulling back the universal small map:

$$\begin{array}{ccc} \dot{\mathsf{U}} & \longrightarrow & \dot{\mathsf{V}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathsf{U} & \longrightarrow & \mathsf{V} \end{array}$$

(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

Since $\mathsf{Fib}(-)$ is stable under pullback, the lower square below is a pullback.

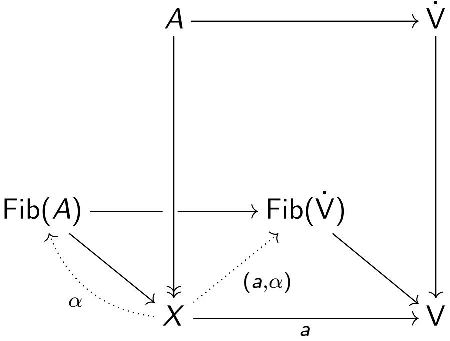


Since $\mathsf{U} = \mathsf{Fib}(\dot{\mathsf{V}})$, there is a section of $\mathsf{Fib}(\dot{\mathsf{U}})$ (namely Δ_U).

So $\dot{\mathsf{U}} \to \mathsf{U}$ is a fibration.

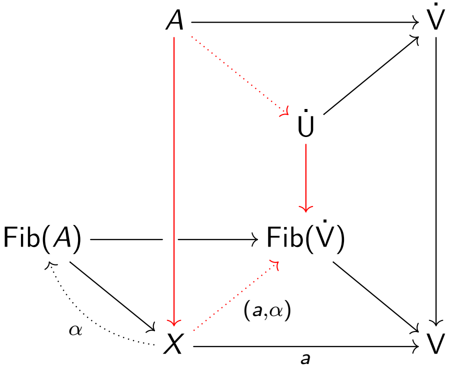
(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

A fibration structure α on a small map $A \to X$ then gives rise to a factorization (a, α) of its classifying map $a : X \to \mathsf{V}$.



(i) A universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$

A fibration structure α on a small map $A \to X$ gives rise to a factorization (a, α) of its classifying map $a : X \to \mathsf{V}$,



which then classifies it as a fibration, since $\mathsf{Fib}(\dot{\mathsf{V}}) = \mathsf{U}$.

(i) The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ in type theory

The type of fibration structures $\mathsf{Fib}(A)$ is an example of type-theoretic thinking.

It can be constructed as the "type of proofs that A is a fibration" using the *propositions-as-types* idea (as explained in Emily Riehl's Topos Colloquium).

A fibration on X is then a pair (A, α) consisting of a small family $A : X \to \mathsf{V}$ together with a proof $\alpha : \mathsf{Fib}(A)$ that A is a fibration.

The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is therefore

$$\mathsf{U} \; = \; \sum_{A:\mathsf{V}} \mathsf{Fib}(A)\,,$$

$$\dot{\mathsf{U}} \; = \; \sum_{(A,\alpha):\mathsf{U}} A\,.$$

(i) The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ in type theory

Recall that the stability of $\mathsf{Fib}(A)$ under pullback required

$$f^*\mathsf{Fib}(A) = \mathsf{Fib}(f^*A) \qquad (*)$$

for any $f : Y \to X$. Indeed, we have a map

$$\mathsf{Fib} : \mathsf{V} \longrightarrow \mathsf{V}$$

for which

$$\mathsf{Fib}(A) = \mathsf{Fib} \circ A\,.$$

So $(*)$ is as simple as:

$$\begin{aligned} f^*\mathsf{Fib}(A) &= \mathsf{Fib}(A) \circ f \\ &= (\mathsf{Fib} \circ A) \circ f \\ &= \mathsf{Fib} \circ (A \circ f) \\ &= \mathsf{Fib}(A \circ f) \\ &= \mathsf{Fib}(f^*A)\,. \end{aligned}$$

(ii) Univalence

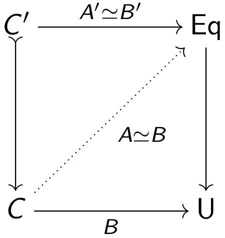
The universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ is *univalent* if the type of (based) equivalences $\mathsf{Eq} \to \mathsf{U}$ is a trivial fibration.

(Once we have the QMS this will imply

$$\mathsf{Id}(A, B) \simeq \mathsf{Eq}(A, B)$$

by the interpretation of Id_{U} as the pathspace $\mathsf{U}^{\mathbb{I}}$.)

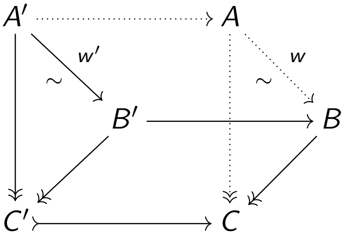
That $\mathsf{Eq} \to \mathsf{U}$ is in TFib means it has the RLP against $\mathcal{C}$:



(ii) Univalence

Definition (EEP)

The *equivalence extension property* says that weak equivalences extend along cofibrations $C' \rightarrowtail C$ as follows: given fibrations $A' \twoheadrightarrow C'$ and $B \twoheadrightarrow C$ and a weak equivalence $w' : A' \simeq B'$, where $B' = C' \times_C B$,



there is a fibration $A \twoheadrightarrow C$ and a weak equivalence $w : A \simeq B$, which pulls back to w'.

(ii) Univalence

Voevodsky proved this for simplicial sets and Kan fibrations, to give the following.

Theorem (Voevodsky)

There is a universal small Kan fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$ *in simplicial sets that is univalent.*

Coquand later gave a constructive proof for cubical sets, using type theoretic reasoning.

We have adapted Coquand's proof to a new homotopical one that holds in many QMCs (without using 3-for-2).

(iii) U is fibrant

From univalence, we can show that the base object U is fibrant.

Theorem
The universe U *is fibrant.*

Voevodsky proved this directly for Kan simplicial sets using *minimal fibrations*, which are specific to that setting.

Shulman gave a general proof from univalence, but it uses 3-for-2 for $\mathcal{W}$, and so cannot be used here.

Coquand gave a proof from univalence that avoids 3-for-2, using a type theoretic reduction of fibrancy to *Kan composition*.

We have a new general proof from univalence that avoids 3-for-2.

(iii) U is fibrant

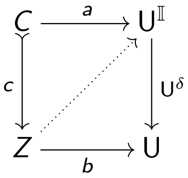
It suffices to show:

Proposition

The evaluation at an endpoint $\mathsf{U}^{\mathbb{I}} \longrightarrow \mathsf{U}$ *is a trivial fibration.*

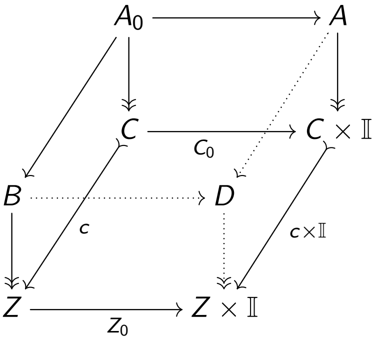
Proof.

We need to solve the following filling problem for any cofibration c.



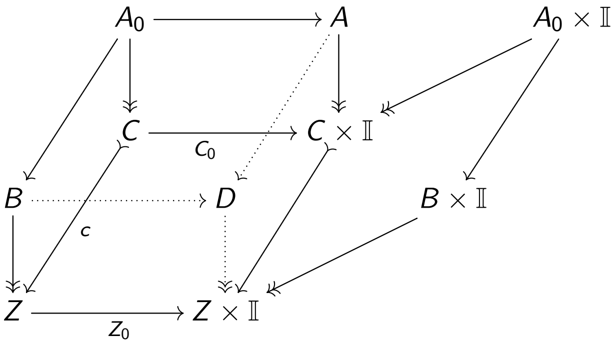
(iii) U is fibrant

Transposing by $\mathbb{I}$ and using the classifying property of U gives the following equivalent problem.



(iii) U is fibrant

Now apply the functor $(-) \times \mathbb{I}$ to the left face to get:



(iii) U is fibrant

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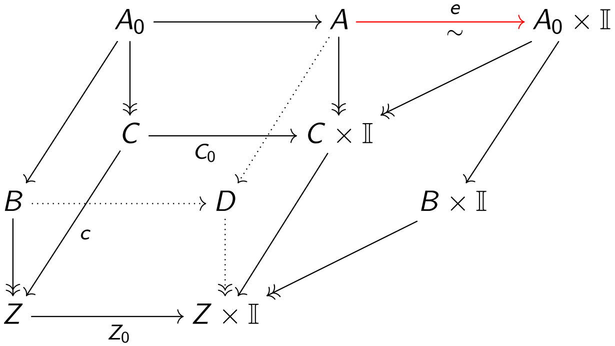
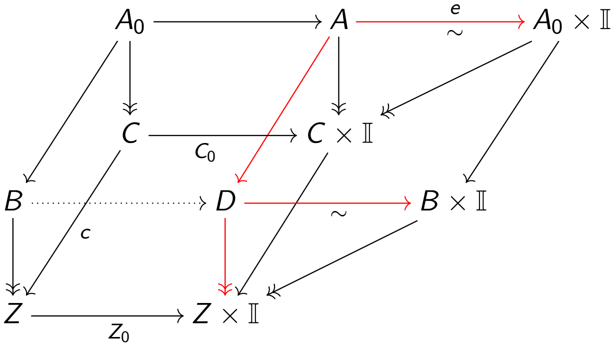


There is a weak equivalence $e : A \xrightarrow{\sim} A_0 \times \mathbb{I}$, to which we can apply the EEP.

(iii) U is fibrant

Now apply the functor $(-) \times \mathbb{I}$ to the left face to get:



There is a weak equivalence $e : A \simeq A_0 \times \mathbb{I}$, to which we can apply the EEP. This produces the required fibration $D \twoheadrightarrow Z \times \mathbb{I}$. □

(iv) From fibrancy of U to 3-for-2

Finally, we can apply the following.

Proposition (Sattler)

The weak equivalences satisfy 3-for-2 if the fibrations extend along the trivial cofibrations.

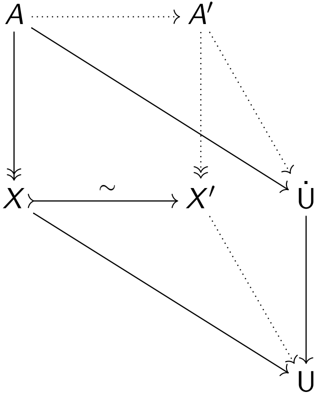
$$
\begin{array}{ccc}
A & \dashrightarrow & A' \\
\downarrow\!\!\downarrow & \lrcorner & \vdots\!\downarrow\!\!\downarrow \\
X & \underset{\sim}{\rightarrowtail} & X'
\end{array}
$$

This is called the *fibration extension property*.

(iv) From fibrancy of U to 3-for-2

Lemma

Given a universal fibration $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$*, the FEP holds if* U *is fibrant.*



□

Discussion: The big picture

There are currently two very different ways to model HoTT:

Logical/Type-Theoretic/Syntactic: In HOL or extensional DTT, axiomatize (what we called) a premodel. Translate the language of HoTT (intensional MLTT) into the resulting axiomatic theory.

Geometric/Homotopical/Semantic: Use a special kind of QMC (roughly, a *type-theoretic model topos* à la Shulman) to interpret intensional MLTT, using a univalent universal fibration.

I did not explain how to actually carry out either of these interpretations. I just described the basic set-up for each. In each case, a lot more work is required to actually give a sound interpretation, i.e. a *model*.

Discussion: The big picture

What I *did* do in this talk was show how to turn a “logical” model into a “geometric” one.

More precisely, we showed how to turn a *presentation* of a logical model (described here as a premodel) into a *presentation* of a geometric model (a QMS).

A missing final step in the comparison would now be something like the following.

Theorem (-ish)

The resulting geometric model is “equivalent” to the logical one that we started from: the interpretation of HoTT into the QMS validates all the same judgements as the interpretation into the logical theory (maybe up to ...).

Something like this will almost certainly be true, once we get the definitions right.

Discussion: The big picture

This then begs the following:

Question

Which geometric models admit such a logical presentation?

Since the QMCs that we are using as geometric models describe ∞-topoi, we are in effect asking which ∞-topoi admit such a logical presentation.

Moreover, a logical presentation can be described as a certain kind of structured 1-topos, as we saw. So we can also formulate the question entirely semantically: *Which ∞-topoi can be presented in this way by a premodel in a 1-topos?*

My own *guess* would be that only a narrow range of all possible ∞-topoi admit such a description in terms of a structured 1-topos.

Thanks!