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A category of hybrid dynamical systems

What's a hybrid dynamical system?

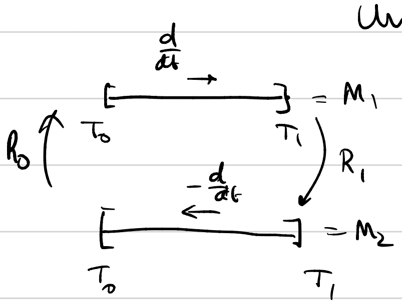
<u>Ex 1</u> "A room with a thermostat": an office in an old building with a window AC unit; it's summer.
There is one variable T = temperature, $T \in [T_0, T_1]$.
Say $T < T_1$; the room is heating up. Say

$$\frac{dT}{dt} = 1 \quad \text{(in some units)}$$

Once T reaches T_1 AC turns on and the room cools:

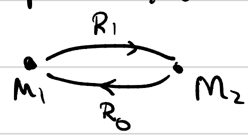
$$\frac{dT}{dt} = -1$$

until T reaches T_0 and then AC turns itself off

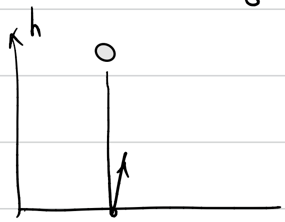


R_1 is a map from $\{T_1\} \subseteq M_1$ to M_2, $R_1(T_1) = T_1 \in M_2$

R_0 is a map from $\{T_0\} \subseteq M_2$ to M_1, $R_0(T_0) = T_0 \in M_2$ $[0, T_1]$



<u>Ex 2</u> Bouncing ball:

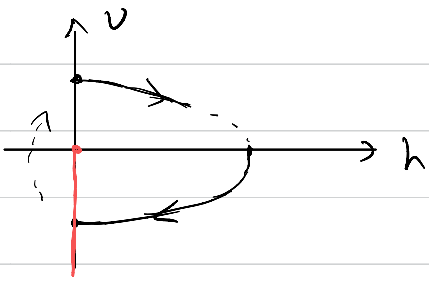


Two variables: $h \geq 0$ (height)
$v \in \mathbb{R}$, velocity

$$\boxed{\frac{dv}{dt} = -g \qquad \frac{dh}{dt} = v}$$

Say collision with the floor is perfectly ellastic, so $E = \frac{mv^2}{2} + mgh$ is conserved. During collision w. floor v changes to $-v$.

$$h = \frac{1}{g}\left(\frac{E}{m} - \frac{v^2}{2}\right)$$



$M = \{(h, v) \in \mathbb{R}^2 \mid h \geq 0\}$ $\quad X = v\frac{\partial}{\partial x} - g\frac{\partial}{\partial v}$

$R: \{0\} \times (-\infty, 0] \longrightarrow M, \quad R(0, v) = (0, -v)$

"a partial map"

<u>Definition</u> A hybrid dynamical system consists of the following data:

- A (directed multi) graph $A = \{A_1 \rightrightarrows A_0\}$
- $\forall$ node/vertex $a \in A_0$ of A a manifold with corners R_a and a vector field $X_a : R_a \to TR_a$
- $\forall$ edge/arrow $a \xrightarrow{\gamma} b$ of A a relation $R_\gamma : R_a \to R_b$, i.e. a subset $R_\gamma \subseteq R_a \times R_b$.

  R_γ is called a <u>reset</u> relation

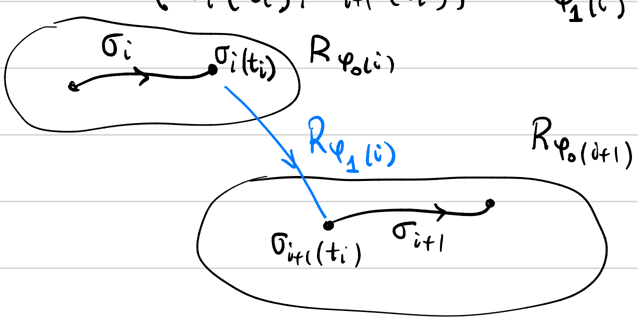
<u>Definition</u> An <u>execution</u> of a hybrid dynamical system $(A, \{(R_a, X_a)\}_{a \in A_0}, \{R_\gamma\}_{\gamma \in A_1})$ is 1) an increasing sequence $\{t_i\}_{i \in \mathbb{N}}$ of real numbers ($i \in \mathbb{Z}$ is also used)

2) a function $\varphi_0 : \mathbb{N} \to A_0$

3) a function $\varphi_1 : \mathbb{N} \to A_1$ compatible with φ_0 : $\varphi_0(i) \xrightarrow{\varphi_1(i)} \varphi_0(i+1)$

4) an integral curve $\sigma_i : [t_{i-1}, t_i] \to R_{\varphi_0(i)}$ of $X_{\varphi_0(i)} : R_{\varphi_0(i)} \to TR_{\varphi_0(i)}$ so that

$(\sigma_i(t_i), \sigma_{i+1}(t_i)) \in R_{\varphi_1(i)}$



How do we turn hybrid dynamical systems into a category?

Well, how do we turn continuous time dynamical systems into a category?

A continuous time dynamical system is a pair (M, X) where M is a manifold and $X : M \to TM$ a vector field. A morphism from (M, X) to (N, Y) is a smooth map $f : M \to N$ so that $Tf \circ X = Y \circ f$:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"'] & N \arrow[u, "Y"'] \end{tikzcd}$$

commutes.

Y & X are <u>f-related</u> or Y & X are <u>semi-conjugate</u>

We get a category DS : objects $(M, X : M \to TM)$

morphisms $Hom_{DS}((M,X),(N,Y)) = \{ f : M \to N \mid Tf \circ X = Y \circ f \}$.

Remarks

1. An integral curve $\gamma: I \to M$ of a vector field $X: M \to TM$ is a smooth map so that $T\gamma\left(\frac{d}{dt}\right) = X \circ \gamma$, ie.
$\gamma: (I, \frac{d}{dt}) \longrightarrow (M, X)$ is a morphism in DS.

2. There is a forgetful functor $U: DS \to Man$, $U\left((M,X) \xrightarrow{f} (N,Y)\right) = M \xrightarrow{f} N$.
U doesn't seem to have any lifting properties, but...
The assignement $M \longmapsto \mathcal{X}(M)$ = vector space of vector fields on M does extend to a (lax 2-) functor $\mathcal{X}: Man \longrightarrow Rel\,Vect$
$Rel\,Vect$ = the 2-category of vector spaces, linear relations and inclusions
$\mathcal{X}(M \xrightarrow{f} N) := \{(X,Y) \in \mathcal{X}(M) \times \mathcal{X}(N) \mid Tf \circ X = Y \circ f\}$
Given $M \xrightarrow{f} N \xrightarrow{g} Q$, $\mathcal{X}(g) \circ \mathcal{X}(f) \subseteq \mathcal{X}(g \circ f)$.

Finally, DS (dynamical systems) is the "category of elements" for $\mathcal{X}: Man \to Rel\,Vect$ since
$$\mathrm{Hom}_{DS}\left((M,X), (N,Y)\right) = \{ f: M \to N \mid (X,Y) \in \mathcal{X}(f)\}.$$
Here I am thinking $\mathcal{X}(f)\,X = Y$

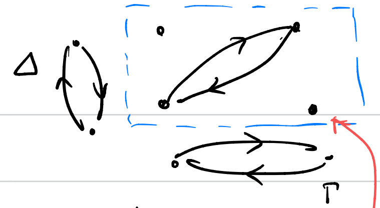
Let's try something similar for hybrid systems.

Definition (provisional) Let $Rel\,Man$ = category of manifolds and (set-theoretic) relations
A *hybrid phase space* ("hybrid manifold") is a graph A together with a map of graphs $A \longrightarrow U(Rel\,Man)$
($U(Rel\,Man)$ = the graph underlying the category $Rel\,Man$)

The issue of defining morphisms aside, is this right?

Ex: Two rooms with two thermostats: phase space should be the product
$\left(\varphi: \quad \to U(Rel\,Man)\right) \times \left(\psi: \quad \longrightarrow U(Rel\,Man)\right) = \varphi \times \psi: \quad \times \quad \to U(Rel\,Man)^2$

But $(\Gamma_1 \rightrightarrows \Gamma_0) \times (\Delta_1 \rightrightarrows \Delta_0) = (\Gamma_1 \times \Delta_1 \rightrightarrows \Gamma_0 \times \Delta_0)$ and so



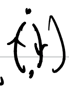
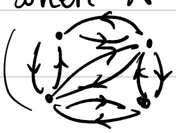
Each dot here is the square $[T_0, T_1] \times [T_0, T_1]$.

This says: the two AC's have to turn on / turn off simultaneously!
But this is not what happens in real life.

Solution The adjunction Free : Graph $\rightleftarrows$ Cat : U

So try: a hybrid phase space is a functor

$$\varphi : \mathrm{Free}(A) \to \mathrm{RelMan}, \text{ where } A \text{ is a graph.}$$

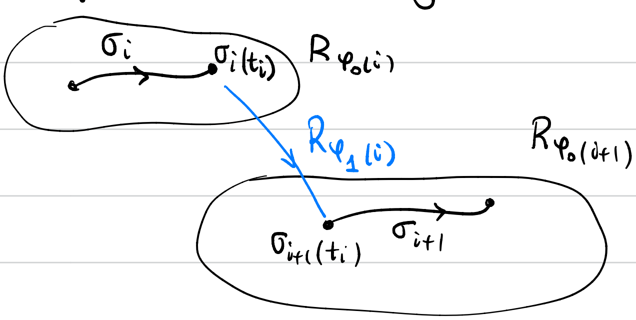
Check Free () $\times$ Free () = Free (), which seems right.

(This is similar to parallel composition in labelled transition systems).

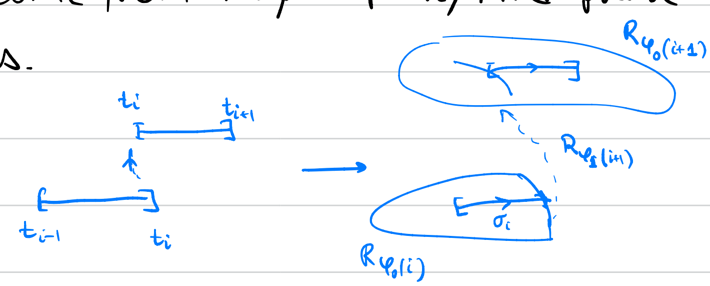
Now drop graph and Free:

> A hybrid phase space is a functor $F : A \to \mathrm{RelVect}$ where A is an indexing category

Morphisms? Recall integral curves of vector fields are smooth maps so executions should come from maps of hybrid phase spaces.



This means:



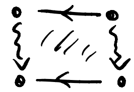
Definition A morphism/map from $F : A \to \mathrm{RelMan}$ to $G : B \to \mathrm{RelMan}$ is a functor $\varphi : A \to B$ and a "natural transformation" $\sigma : F \Rightarrow G \circ \varphi$, $a \mapsto \left(F(a) \xrightarrow{\sigma_a} G(\varphi(a)) \right)$, σ_a = a <u>smooth function</u>
so that $\forall \; a \xrightarrow{\gamma} a'$ in A

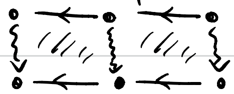
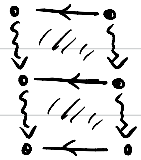
$$\begin{tikzcd} F(a) \arrow[r, "\sigma_a"] \arrow[d, squiggly, "F(\gamma)"'] \arrow[dr, phantom, "////"] & G(\varphi(a)) \arrow[d, squiggly, "G(\varphi(\gamma))"] \\ F(a') \arrow[r, "\sigma_{a'}"'] & G(\varphi(a')) \end{tikzcd}$$

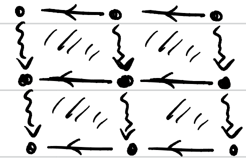
ie $(\sigma_a \times \sigma_{a'})(F(\gamma)) \subseteq G(\varphi(\gamma))$

What's going on?

Manifolds, smooth functions and relations form a double category $\mathrm{RelMan}^{\square}$.

"Recall" A double category has objects, two types of 1-arrows "vertical" and "horizontal" and tiles/squares:  . Tiles can be composed vertically and horizontally:

  .

And the two compositions commute: 

There is a notion of a functor between double categories: tiles go to tiles and the two compositions are preserved.

An ordinary 1-category is a double category:

$$a \xrightarrow{\gamma} b \quad = \quad \begin{tikzcd} a \arrow[r, "\gamma"] \arrow[d, squiggly, "id_a"'] \arrow[dr, phantom, "////"] & b \arrow[d, squiggly, "id_b"] \\ a \arrow[r, "\gamma"'] & b \end{tikzcd}$$

There are two kinds of natural transformations: "vertical" and "horizontal"

Definition The category HyPh of hybrid phase spaces:

objects: double functors $F : A \to RelMan^{\square}$ where A is a (small) category and $RelMan^{\square}$ the double category of manifolds, smooth functions and relations.

Morphisms: "2-commuting" triangles

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & B \arrow[dl, "G"] \\ & RelMan^{\square} & \end{tikzcd}$$

There is a forgetful functor $\mathcal{U} : HyPh \to Man$: given $F : A \to RelMan^{\square}$

$\mathcal{U}(F) = \coprod_{a \in A_0} F(a)$. (and

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\sigma}{\Rightarrow}"] & B \arrow[dl, "G"] \\ & RelMan^{\square} & \end{tikzcd}$$

induce

$$\mathcal{U}(\varphi, \sigma) : \coprod_{a \in A_0} F(a) \longrightarrow \coprod_{b \in B_0} G(b)$$

Compose $\mathcal{U} : HyPh \to Man$ with $\mathcal{X} : Man \to RelVect$ ($M \mapsto \mathcal{X}(M) =$ space of vector fields on M)

and take the category of elements of $\mathcal{X} \circ \mathcal{U} : HyPh \to RelVect$.

You get HyDS, the category of hybrid dynamical systems.

$$
\begin{tikzcd}
\mathrm{HyDS} \arrow[r, "\text{forget resets}"] \arrow[d, "\text{forget v.f.}"'] & \mathrm{DS} \arrow[d, "\text{forget v.f.}"] \\
\mathrm{HyPh} \arrow[r, "\text{forget resets}"'] & \mathrm{Man}
\end{tikzcd}
$$

Now we can start constructing hybrid open systems...

One possible drawback of the set up: we can easily vary vector fields but resets are fixed — They are baked into the definition of a hybrid phase space.

One may wish to vary resets as well — James Schmidt did This in his PhD Thesis in 2019.