From 2-rigs to λ-rings

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Outline of talk

- Report on Baez, Moeller, T. *Schur functors and categorified plethysm*, https://arxiv.org/abs/2106.00190

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  item Outline
  1. Plethories
  2. 2-plethories
  3. Decategorify the simplest 2-plethory to get the most beautiful object

What is a plethory?

"Classically" (Tall-Wraith, Stacey-Whitehouse, Borger-Wieland), a **plethory** consists of a (commutative) ring B

- Equipped with a lift Φ as in

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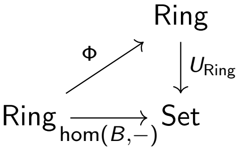

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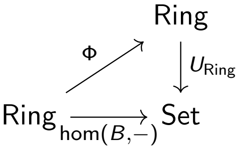

- Together with a comonad structure on Φ.
- Really, the same thing as a right adjoint comonad on Ring! (notes 1, 2)

Concrete description of lift

More concretely: the lift Φ in

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amounts to putting a ring structure on hom-sets $\hom(B, R)$, naturally in R:

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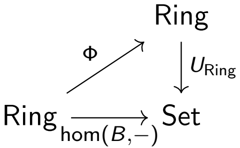

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$$\mu : B \to B \otimes B \quad \text{“comultiplication”}$$

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  1. Comultiplication $\mu : B \to B \otimes B$,
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  3. Co-zero $o : B \to \mathbb{Z}$ (map to initial ring)
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"Biring" B: co-ring object in $(+)$-monoidal category of rings.

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$$\left(\text{Ring} \xrightarrow{\Phi} \text{Ring} \xrightarrow{U} \text{Set}\right) \cong (\hom(B,-) : \text{Ring} \to \text{Set})$$

Concretely, what is a comonad structure $\delta : \Phi \to \Phi\Phi, \varepsilon : \Phi \to \mathbf{1}$?

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Theorem: $\Phi \xrightarrow{\delta} \Phi\Phi$ is coassociative iff plethysm is associative (notes 3, 4).

Example

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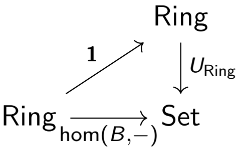

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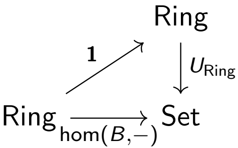

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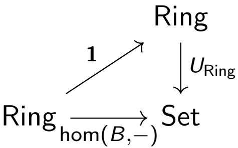

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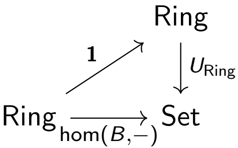

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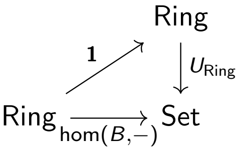

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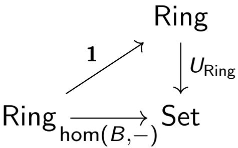

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- Co-zero and co-one are the maps $B \to \mathbb{Z}$ sending $x \mapsto 0, 1$.

Plethysm for $\mathbb{Z}[x]$

Here $\Phi = \mathbf{1}_{\mathsf{Ring}}$. The map $h : B \to \Phi B$ is the identity map:

$$\begin{array}{ccccc} \mathbb{Z}[x] & \to & \Phi\mathbb{Z}[x] & \stackrel{\sim}{\to} & \hom(\mathbb{Z}[x], \mathbb{Z}[x]) \\ q(x) & \mapsto & q(x) & \mapsto & (x \mapsto q(x)) \end{array}$$

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$$\mathbb{Z}[x] \times \mathbb{Z}[x] \to \mathbb{Z}[x]$$

$$(q(x), p(x)) \mapsto p(q(x))$$

"Plethysm is substitution monoidal structure".

Definition of 2-rig

Categorifying the notion of rig: for us, a **2-rig** is

- A Vect-enriched category $\mathcal{C}$,
- Closed under *absolute* colimits (giving 2-additive structure)
  1. Biproducts = direct sums $A \oplus B$,
  2. Idempotent splittings: every idempotent $e : A \to A$ factors as a retraction $r : A \to B$ followed by a section $i : B \to A$, so that $e = ir$ and $ri = 1_B$. In that case have a coequalizer

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- Equipped with a symmetric monoidal structure $\otimes$ in the Vect-enriched sense (giving 2-multiplicative structure).

Here $A \otimes B$ automatically preserves absolute colimits in each of the separate arguments A, B, i.e., 2-distributivity is automatic in this context.

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The free 2-rig on one generator: mimic the construction of the free rig $\mathbb{N}[x]$ on one generator.

Form $\mathbb{N}[x]$ in two steps:

- First form the free commutative (multiplicative) monoid on one generator x (monomials x^n).
- Then form the free commutative (additive) monoid on that (polynomials $a_0 + a_1 x + \ldots + a_n x^n$ with coefficients in $\mathbb{N}$).

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Analogously, to form the free 2-rig Vec$[x]$ on one generator:

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  3. Close up under idempotent splittings. If $\mathrm{char}(k) = 0$, one gets arbitrary functors

$$\mathbb{P} \to \mathrm{Vec}$$

of finite support.

Example

In $k[S_2]$, let σ be the transposition. Define idempotent maps $e_-, e_+ : k[S_2] \to k[S_2]$ given by multiplying by $\frac{1}{2}(1 \pm \sigma)$, e.g.,

$$e_-^2 = \left(\frac{1-\sigma}{2}\right)^2 = \frac{1-2\sigma+\sigma^2}{4} = \frac{2-2\sigma}{4} = \frac{1-\sigma}{2}.$$

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If $\mathcal{C}$ is any 2-rig, with an object V, the 2-rig map $\mathrm{Vec}[x] \to \mathcal{C}$ that sends x to V also sends the retract E_- to a retract

$$V^{\otimes 2} \to E_-(V) \to V^{\otimes 2},$$

usually denoted $E_-(V) = \Lambda^2(V)$.

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usually denoted $E_-(V) = \Lambda^2(V)$. If E_+ is the retract corresponding to $e_+ = \frac{1}{2}(1+\sigma)$, then we similarly have a retract $E_+(V)$ of $V^{\otimes 2}$, usually denoted $E_+(V) = S^2(V)$.

2-birig structure

Analogous to the biring structure on $\mathbb{Z}[x]$, there is a 2-birig structure on $\mathrm{Vec}[x]$:

- (2-)coproducts of 2-rigs are like tensor products. If $\mathcal{C}$ and $\mathcal{D}$ are 2-rigs, then their coproduct $\mathcal{C} \boxtimes \mathcal{D}$ is the closure under biproducts, idempotent splittings of the tensor $C \otimes D$, where

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- We have $\mathsf{Vec}[x_1] \boxtimes \mathsf{Vec}[x_2] \simeq \mathsf{Vec}[x_1, x_2]$. This is the same as the category of finitary functors

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- Co-multiplication $\mathrm{Vec}[x] \to \mathrm{Vec}[x_1, x_2]$ sends x to $x_1 \otimes x_2$.
- 2-birig co-zero, co-one $\mathrm{Vec}[x] \to \mathrm{Vec}$ send x to $0, k$.

2-plethysm structure

$$
\begin{array}{ccc}
 & & \mathsf{2Rig} \\
 & \overset{\mathbf{1}}{\nearrow} & \big\downarrow U_{\mathsf{2Rig}} \\
\mathsf{2Rig} & \xrightarrow[\hom(B,-)]{} & \mathsf{Cat}.
\end{array}
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$\delta : \mathbf{1} \to \mathbf{11}$ corresponds to a 2-rig map $h = \mathrm{id} : \mathsf{Vec}[x] \to \mathbf{1}\mathsf{Vec}[x]$.
This corresponds in turn to the canonical 2-rig map

$$\begin{array}{rcl} \mathsf{Vec}[x] & \to & \mathsf{2Rig}(\mathsf{Vec}[x], \mathsf{Vec}[x]) \\ Q & \mapsto & (x \mapsto Q) \end{array}$$

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 & & x^{\otimes n} \mapsto Q^{\otimes n} \\
 & & \sum_n a_n x^{\otimes n} \mapsto \sum_n a_n Q^{\otimes n} \\
 & & \left(A \xrightarrow{r} E \xrightarrow{i} A\right) \mapsto (A \circ Q \to E \circ Q \to A \circ Q)
\end{array}
$$

2-plethysm structure

De-currying, we obtain a functor (2-plethysm)

$$\mathrm{Vec}[x] \times \mathrm{Vec}[x] \to \mathrm{Vec}[x]$$

$$(Q, E) \mapsto E \circ Q$$

Theorem: The 2-plethysm functor is associative up to coherent isomorphism. It is part of a (plethystic) monoidal category structure on Vec[x]. (Compare "substitution product" of Joyal species.)

Decategorification

The "trivial" 2-plethory $\mathbf{1} : 2\mathrm{Rig} \to 2\mathrm{Rig}$ descends, through decategorification, to a highly nontrivial rig-plethory $\mathrm{Rig} \to \mathrm{Rig}$!

General idea: if B is a 2-rig, and if $H(B)$ is the set of isomorphism classes of objects of B, then $\otimes$ and $\oplus$ on B induce a rig structure on $H(B)$. This rig is denoted $J(B)$.

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But what do we do with the 2-cells of 2Rig?

Definition: If $\mathcal{C}$ is a 2-category (a Cat-enriched category), then $\mathcal{C}_{\mathrm{ho}}$ is the ordinary (Set-enriched) category obtained by applying

$$\mathrm{Cat} \stackrel{\mathrm{core}}{\to} \mathrm{Gpd} \stackrel{\pi_0}{\to} \mathrm{Set}$$

to the Cat-valued homs of $\mathcal{C}$.

Decategorification

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Proposition: If C is a category, regarded as an object of $\mathsf{Cat}_{\mathsf{ho}}$, then

$$\hom(1, -) : \mathsf{Cat}_{\mathsf{ho}} \to \mathsf{Set}$$

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Corollary: If $\mathcal{C}$ is a 2-category, and c, d are objects, then

$$\hom(1, \mathcal{C}(c, d)) = \mathcal{C}_{\mathrm{ho}}(c, d)$$

regarding the category $\mathcal{C}(c, d)$ as belonging to $\mathrm{Cat}_{\mathrm{ho}}$.

Decategorification

Proposition: There is a product-preserving lift J of the bottom composite in

$$\begin{array}{ccccc} & & & & \mathsf{Rig} \\ & & {}^{J}\nearrow & & \downarrow U_{\mathsf{Rig}} \\ \mathsf{2Rig}_{\mathsf{ho}} & \xrightarrow[U_{\mathsf{ho}}]{} & \mathsf{Cat}_{\mathsf{ho}} & \xrightarrow[\hom(1,-)]{} & \mathsf{Set} \end{array}$$

Namely, if R is a 2-rig, then $U_{\mathsf{ho}}(R)$ is a rig object in $\mathsf{Cat}_{\mathsf{ho}}$, and the functor $\hom(1,-)$ is product-preserving (hence preserves rig objects). Hence the set

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carries rig structure. This rig is $J(R)$. We let $\Lambda_+ = J(\mathsf{Vec}[x])$.

$$H(R) \cong \hom(1, \mathsf{2Rig}(\mathsf{Vec}[x], R)) = \mathsf{2Rig_{ho}}(\mathsf{Vec}[x], R)$$

Λ_+ is a birig

Theorem: $J : 2\mathrm{Rig}_{\mathrm{ho}} \to \mathrm{Rig}$ preserves copowers of $\mathrm{Vec}[x]$. (note 5)

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This result allows us to define a birig structure on $\Lambda_+ = J(\mathrm{Vec}[x])$. For example, start with 2-coaddition

$$\alpha : \mathrm{Vec}[x] \to \mathrm{Vec}[x] \boxtimes \mathrm{Vec}[x]$$

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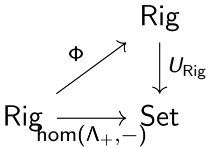
Compose:

$$J(\mathrm{Vec}[x]) \stackrel{J(\alpha)}{\to} J(\mathrm{Vec}[x] \boxtimes \mathrm{Vec}[x]) \cong J(\mathrm{Vec}[x]) \otimes J(\mathrm{Vec}[x])$$

$$\text{co-add} : \Lambda_+ \to \Lambda_+ \otimes \Lambda_+$$

Rig-plethory structure on Λ_+

The birig structure on Λ_+ provides a lift Φ



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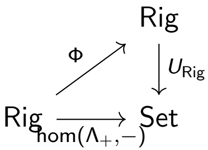
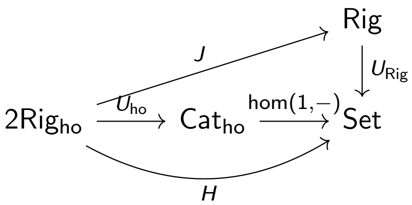


We want a comonad structure on Φ: a comultiplication $\delta : \Phi \to \Phi\Phi$ and counit $\varepsilon : \Phi \to \mathbf{1}_{\mathsf{Rig}}$.

Recall that $U\Phi \to U\Phi\Phi$ amounts to structure map

$$h : U\Lambda_+ \to \mathrm{Rig}(\Lambda_+, \Lambda_+).$$

Rig-plethory structure on $\Lambda_+ = J(\mathsf{Vec}[x])$



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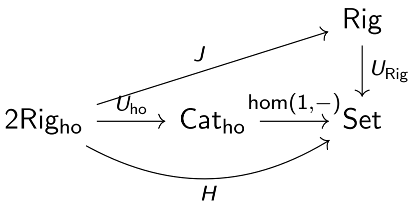

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Compose:

$$UJ(\mathsf{Vec}[x]) \cong 2\mathsf{Rig}_{\mathrm{ho}}(\mathsf{Vec}[x], \mathsf{Vec}[x]) \to \mathsf{Rig}(J(\mathsf{Vec}[x]), J(\mathsf{Vec}[x]))$$

$$U\Lambda_+ \to \mathsf{Rig}(\Lambda_+, \Lambda_+)$$

Theorem: The (rig) map

$$U\Lambda_+ \to \mathrm{Rig}(\Lambda_+, \Lambda_+)$$

gives rise to an associative plethystic multiplication

$$U\Lambda_+ \times U\Lambda_+ \to U\Lambda_+,$$

compatibly with the birig structure (note 4), so that it defines a coassociative transformation $\delta : \Phi \to \Phi\Phi$ that is part of a rig-plethory (a right adjoint comonad on Rig).

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The ringification

$$\Lambda = \mathbb{Z} \otimes_{\mathbb{N}} \Lambda_+$$

(the Grothendieck ring $K_0(\mathrm{Vec}[x])$) similarly carries a canonical plethory structure, i.e., there is a canonical lift Φ : Ring $\to$ Ring of $\hom(\Lambda, -)$: Ring $\to$ Set with a comonad structure. (note 6)

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A λ-*ring* is then a Φ-coalgebra. (notes 7, 8)

Thank you!

Notes

1. Or a left adjoint monad on Ring. (String diagrams show that the left adjoint of a comonad carries a monad structure, and dually the right adjoint of a monad carries a comonad structure; the back-and forths are mutually inverse.) Moreover, in this case the category of coalgebras of the comonad is equivalent to the category of algebras of the monad: the forgetful functor from either is both monadic and comonadic.

2. If Φ is a right adjoint, then so is $U\Phi$, and any right adjoint to Set is representable: $U\Phi \cong \mathrm{hom}(B, -)$. Conversely, by an adjoint lifting theorem, any lift of a representable through $U : \mathrm{Ring} \to \mathrm{Set}$ must be a right adjoint. The same applies to any monadic category $U : C \to \mathrm{Set}$ in place of $U : \mathrm{Ring} \to \mathrm{Set}$.

Notes

3. Of course there is also the counit $\varepsilon : \Phi \to \mathbf{1}$. We have $U\mathbf{1} \cong \text{hom}(\mathbb{Z}[x], -)$. Thus

$$\begin{array}{rcl} U\Phi & \stackrel{U\varepsilon}{\to} & U\mathbf{1} \\ \text{hom}(B, -) & \to & \text{hom}(\mathbb{Z}[x], -) \\ \mathbb{Z}[x] & \to & B. \end{array}$$

The composite

$$1 \stackrel{\lceil x \rceil}{\to} U\mathbb{Z}[x] \to UB$$

picks out an element ι of UB. This element is a unit for plethysm iff ε obeys the comonad counit equations.

Notes

4. $H : U\Phi \to U\Phi\Phi$ corresponding to $h : B \to \Phi B$ is of the form $H = U\delta : U\Phi \to U\Phi\Phi$, iff H preserves the ring operations (induced from biring structure). E.g., here preservation of multiplication:

$$\begin{array}{ccc} \hom(B,-)^2 & \xrightarrow{\tilde{h}^2} & \hom(B,\Phi -)^2 \\ {\scriptstyle \text{mult}}\downarrow & & \downarrow{\scriptstyle \text{mult}} \\ \hom(B,-) & \xrightarrow[\tilde{h}]{} & \hom(B,\Phi -) \end{array}$$

In terms of co-multiplication μ, this is equivalent to

$$\begin{array}{ccc} B & \xrightarrow{h} & \Phi B \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \Phi(\mu)} \\ B \otimes B & \xrightarrow{h_{(2)}} & \Phi(B \otimes B) \end{array} \quad , \quad \text{where}$$

$$B \xrightarrow{i_1, i_2} B \otimes B \xrightarrow{h_{(2)}} \Phi(B \otimes B) \; = \; B \xrightarrow{h} \Phi B \xrightarrow{\Phi(i_1),\Phi(i_2)} \Phi(B \otimes B)$$

Notes

5. **Theorem:** Let $n_1, \ldots, n_p$ be natural numbers. If $\mathrm{char}(k) = 0$, then every irrep of $k[S_{n_1} \times \ldots S_{n_p}]$ is a tensor product $\rho_i \otimes \ldots \otimes \rho_p$ of irreps ρ_i of $k[S_{n_i}]$ that are determined uniquely up to isomorphism. $\Box$

Now $J(\mathrm{Vec}[x])$ is a free $\mathbb{N}$-module on isomorphism classes of irreps of symmetric groups, so that $J(\mathrm{Vec}[x])^{\otimes p}$ is a free $\mathbb{N}$-module on p-tuples of such classes. Meanwhile $J(\mathrm{Vec}[x]^{\boxtimes p})$ consists of isomorphism classes of functors $\mathbb{P}^{\times p} \to \mathrm{Vec}$ of finite support. The canonical rig map

$$J(\mathrm{Vec}[x])^{\otimes p} \to J(\mathrm{Vec}[x]^{\boxtimes p})$$

is the $\mathbb{N}$-module map that freely extends the mapping

$$([\rho_1], \ldots, [\rho_p]) \mapsto [\rho_1 \otimes \ldots \otimes \rho_p]$$

Since this mapping is a bijection by the theorem, the canonical rig map is an isomorphism.

Notes

6. The main problem is to define the co-negation co-operation on Λ. This is explained in section 7 of the paper, but in outline, one considers the 2-rig $\mathcal{G}$ of $\mathbb{Z}_2$-graded $\mathrm{Vec}[x]$-objects (C_0, C_1), with the usual symmetry that involves a sign change when permuting homogeneous elements of odd degree. Then $J(\mathcal{G})$ is a rig of pairs $([C_0], [C_1])$ and there is a well-defined rig map

$$\partial : J(\mathcal{G}) \to \Lambda$$

taking $([C_0], [C_1]) \mapsto [C_0] - [C_1]$. Form the 2-rig map $\phi_- : \mathrm{Vec}[x] \to \mathcal{G}$ that takes x to $(0, x)$. This behaves something like a categorified co-negation. Form the rig composite

$$\Lambda_+ = J(\mathrm{Vec}[x]) \stackrel{J(\phi_-)}{\to} J(\mathcal{G}) \stackrel{\partial}{\to} \Lambda$$

and freely extend this rig map to a ring map $\Lambda \to \Lambda$. This gives the desired co-negation on the biring Λ. The proof that this works uses a "categorified Euler formula".

Notes

7. Or again, as in Note 1, the algebras of a left adjoint monad. This is the more usual tack taken (as in Borger-Wieland). The point is that there is an equivalence between left adjoint endofunctors on Ring and birings; since left adjoint endofunctors compose, there is a monoidal structure on Ladj(Ring), which may be transferred across the equivalence

$$\text{Ladj(Ring)} \simeq \text{Biring}$$

to give a monoidal product on Biring, usually denoted $\odot$. Then a plethory may be defined to be a biring B with a $\odot$-monoid structure. In that case, a λ-algebra may be defined to be an algebra for the monad

$$\Psi = \Lambda \odot - : \text{Ring} \to \text{Ring}$$

that is left adjoint to the comonad Φ. Incidentally, this comonad is know as the "big Witt functor W"; W-coalgebras are the same as Λ-algebras.

Notes

8. To be sure, this is not the usual way of presenting λ-rings! One of the virtues however is making explicit the conceptual reason for why the usual examples (virtual differences of group reps, of vector bundles, etc.) form λ-rings: it's because $\mathsf{Vec}[x]$ acts tautologically on any 2-rig via Schur functors, analogous to how the polynomial ring $\mathbb{Z}[x]$ acts tautologically on any ring R by the unique ring map $\mathbb{Z}[x] \to \mathsf{Func}(R, R)$ sending x to 1_R (where the codomain carries the pointwise ring structure).

The usual way is to define a λ-ring as a ring R together a series of operations $\lambda^i : R \to R$ that abstract exterior power operations (exterior powers giving a main example of Schur functors). These λ-operations obey a complicated set of equations which may be found in many texts; we do not reproduce them here.