MMT: A UniFormal Approach to Knowledge Representation

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About Me

My Background

- Areas
  - theoretical foundations
    logic, programming languages foundations of mathematics
  - formal knowledge representation
    specification, formalized mathematics, ontologies, programming
  - scalable applications
    module systems, libraries, system integration
- Methods
  - survey and abstract — understand fundamental concepts
  - relate and transfer — unify different research areas
  - long-term investment — identify stable ideas, do them right
  - modularity and reuse — maximize sharing across languages, tools

My Vision

UniFormal

a universal framework for the formal representation of knowledge

- integrate all domains
  specification, deduction, compuation, mathematics, ...
- integrate all formal systems
  logics, programming languages, foundations of mathematics, ...
- integrate all applications
  theorem provers, library managers, IDEs, wikis ...

My (evolving, partial) solution: MMT framework

- a uniformal knowledge representation framework
  developed since 2006, $\sim 100,000$ loc, ~ 500 pages of publications
- allows foundation-independent solutions
  module system, type reconstruction, ...
  IDE, search, build system, library, ...

`http://uniformal.github.io/`

Motivation

Logic in Computer Science

- $\sim$ 1930: computer science — vision of mechanizing logic
- Competition between multiple logics
  - axiomatic set theory: ZF(C), GBvN, ...
  - λ-calculus:
    - typed or untyped
    - Church-style or Curry-style
  - new types of logic modal, intuitionistic, paraconsistent ,...
- Diversification into many different logics
  - fine-tuned for diverse problem domains far beyond predicate calculus
  - deep automation support decision problems, model finding, proof search, ...
  - extensions towards programming languages

History of Formal Systems

- late 19th century: formal axiomatizations
- $\sim$ 1900: paradoxa in logic, mathematics
- $\sim$ 1920s: vision of mechanizing logic
- $\sim$ 1930s: birth of computer science

Desire for automating

- formal representation
- computation
- logical proof

Universal approach to intertwined problems

Competition between multiple languages

- axiomatic set theory: ZF, GBvN, ...
- type theory: Principia Mathematica, λ-calculus
- new logics: modal, intuitionistic, advanced type systems...

Diversification into many different languages

Selected Major Successes

Verified mathematical proofs

- 2006–2012: Gonthier et al., Feit-Thompson theorem
  170,000 lines of human-written formal logic
- 2003–2014: Hales et. al., Kepler conjecture (Flyspeck)
  $> 5,000$ processor hours needed to check proof

Software verification

- 2004–2010: Klein et al., L4 micro-kernel operating system
  390,000 lines of human-written formal logic
- since 2005: Leroy et al., C compiler (CompCert)
  almost complete, high performance

Knowledge-based Artificial intelligence

- since 1984: Lenat et al., common knowledge (CyC)
  2 million facts in public version
- since 2000: Pease et. al., foundation ontology (SUMO)
  $25,000$ concepts

Future Challenges

Huge potential, still mostly unrealized

Applications must reach much larger scales

- software verification successes dwarfed by practical needs
  internet security, safety-critical systems, . . .
- automation of math barely taken seriously by mathematicians

Applications must become much cheaper

- mostly research prototypes
- usually require PhD in logic
- tough learning curve
- time-intensive formalization

The Dilemma of Fixed Foundations

Each system fixes a logic and/or programming language

- type theories, set theories, first-order logics, higher-order logics, ...
  ACL2, Coq, HOL, Isabelle/HOL, Matita, Mizar, Nuprl, PVS,...
- functional, imperative, inheritance-oriented, soft typing ...
  Axiom, Sage, GAP, Maple, ...
- Foundation-specific results
  contrast to mathematics: foundation left implicit
- All systems mutually incompatible

Exacerbates the other bottlenecks

- Human resource bottleneck
  - no reuse across systems
  - very slow evolution of systems
- Knowledge management bottleneck
  - retrofitting to fixed foundation systems very difficult
    can be easier to restart from scratch
  - best case scenario: duplicate effort for each system

Two Formidable Bottlenecks

Each system requires $\approx$ 100 person-year investment to

- design the foundational logic
- implement it in a computer system
- build and verify a collection of formal definitions and theorems
  e.g., covering undergraduate mathematics
- apply to practical problems

human resource bottleneck

New scales brought new challenges

- no good search for previous results
  reproving can be faster than finding a theorem
- no change management support
  system updates often break previous work
- no good user interfaces far behind software engineering IDEs

knowledge management bottleneck

Example Problems

Collaborative QED Project, 1994

- high-profile attempt at building single library of formal mathematics
- failed partially due to disagreement on foundational logic

Voevodsky's Homotopy Type Theory, since 2012

- high-profile mathematician interested in applying logic
- his first result: design of a new foundation

Multiple 100 person-year libraries of mathematics

- developed over the last $\sim$ 30 years
- overlapping but mutually incompatible major duplication of efforts
- translations mostly infeasible

Hales's Kepler Proof

- distributed over two separate implementations of the same logic
- little hope of merging

Vision

UniFormal

a universal framework for the
formal representation of all knowledge and its semantics
in math, logic, and computer science

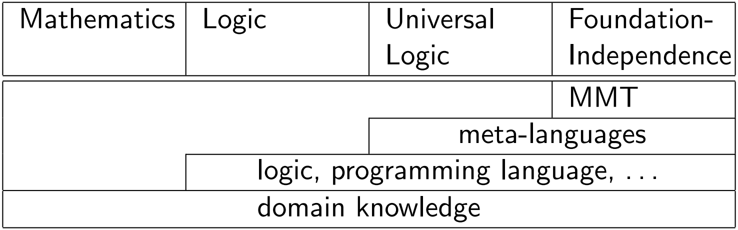
- Avoid fixing languages wherever possible ...
- ... and instantiate them for different languages
- Use formal meta-languages in which to define languages ...
- ... and avoid fixing even the meta-language
- Obtain foundation-independent results
  - Representation languages
  - Soundness-critical algorithms
  - Knowledge management services
  - User-facing applications

MMT as a UniFormal Framework

MMT = Meta-Meta-Theory/Tool

few primitives . . . that unify different domain concepts

- tiny but universal grammar for expressions
  syntax trees with binding
- standardized semantics of identifiers crucial for interoperability
- high-level definitions derivation, model, soundness, . . .
- no built-in logic or type system all algorithms parametric
- theories and theory morphisms for large scale structure
  translation, interpretation, semantics, . . .



Disclaimer: MMT is not a proof assistant

MMT allows

- defining/implementing formal systems
- reasoning in an about them
- building and integrating libraries
- developing language-independent support tools

Why is none of the support tools a proof assistant?

- easy to do, very hard to become competitive
- difficult for student projects
- unclear how to integrate
  - logic-independent methods
  - dedicated logic-specific ones

MMT is Kind of Like

(allowing for a grain of salt)

Isabelle but

- no built-in λ-calculus
- no good proof support (yet)

Twelf but

- users can change the logical framework flexibly
- more knowledge management

QED project but

- actually attempted
- individual proof assistant libraries not integrated yet

Subsume All Aspects of Knowledge

- Narration: informal-but-rigorous math
  needed for human consumption
- Deduction: logic and type systems
  needed for machine understanding
- Computation: data structures and algorithms
  needed for practical applications
- Data: tabulate large sets and functions
  needed for examples, exploration and efficiency

Deduction Data

Narration Computation

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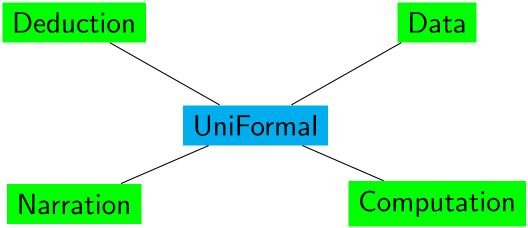
- Data: tabulate large sets and functions

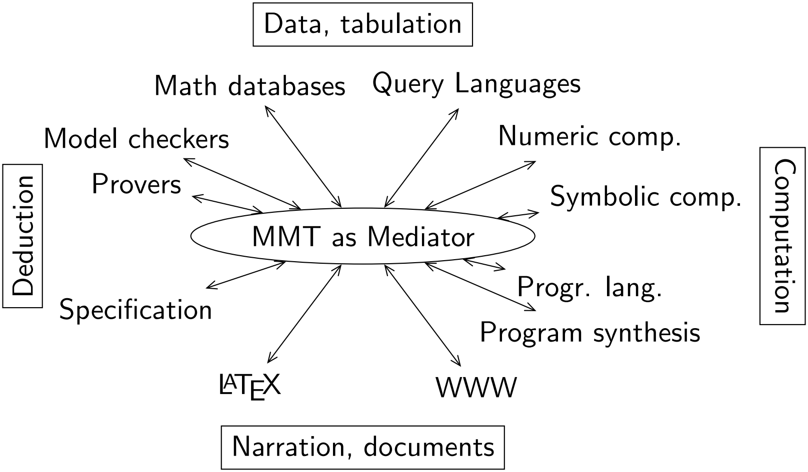
  needed for examples, exploration and efficiency
- Universal representation language

  key to universality, inter-operability



MMT as System Integration Platform

All system interfaces formalized in MMT
→ semantics-aware tool integration while maintaining existing work flows



Basic Concepts

Design principle

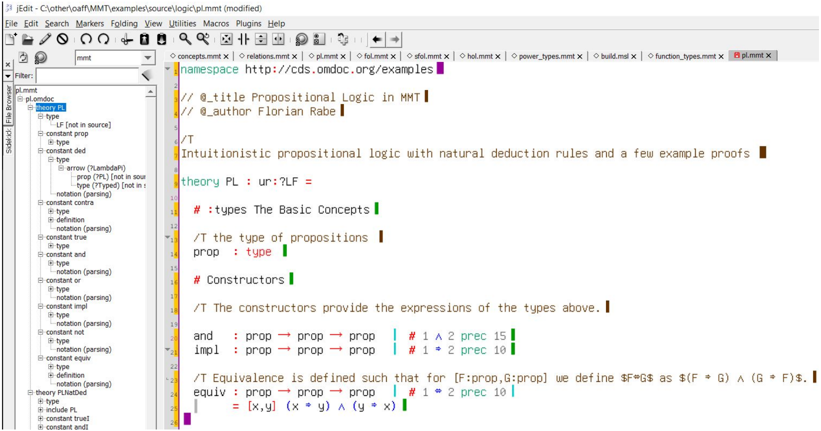
- few orthogonal concepts
- uniform representations of diverse languages

sweet spot in the expressivity-simplicity trade off

Concepts

- theory = named set of declarations
  - foundations, logics, type theories, classes, specifications, ...
- theory morphism = compositional translation
  - inclusions, translations, models, katamorphisms, ...
- constant = named atomic declaration
  - function symbols, theorems, rules, ...
  - may have type, definition, notation
- term = unnamed complex entity, formed from constants
  - expressions, types, formulas, proofs, ...
- typing $\vdash_T s : t$ between terms relative to a theory
  - well-formedness, truth, consequence ...

Example: Propositional Logic in the MMT IDE



Small Scale Example (1)

Logical frameworks in MMT

```
theory LF {
   type
   Pi        # Π V1 . 2                    name[:type][#notation]
   arrow     # 1 → 2
   lambda    # λ V1 . 2
   apply     # 1 2
}
```

Logics in MMT/LF

```
theory Logic: LF {
   prop : type
   ded  : prop → type  # ⊢ 1                judgments-as-types
}
theory FOL: LF {
  include Logic
  term       : type                          higher-order abstract syntax
  forall     : (term → prop) → prop  #  ∀ V1 . 2
}
```

Small Scale Example (2)

FOL from previous slide:

```
theory FOL: LF {
  include Logic
  term         : type
  forall       : (term → prop) → prop  #  ∀ V1 . 2
}
```

Proof-theoretical semantics of FOL

```
theory FOLPF: LF {
  include FOL
                                              rules are constants
  forallIntro  : ΠF:term→prop.
                   (Πx:term.⊢ (F x)) → ⊢ ∀(λx:term.F x)
  forallElim   : ΠF:term→prop.
                   ⊢ ∀(λx:term.F x) → Πx:term.⊢ (F x)
}
```

Small Scale Example (3)

FOL from previous slide:

```
theory FOL: LF {
  include Logic
  term          : type
  forall        : (term → prop) → prop  #  ∀ V1 . 2
}
```

Algebraic theories in MMT/LF/FOL:

```
theory Magma : FOL {
  comp : term → term → term # 1 ∘ 2
}
theory SemiGroup          : FOL {include Magma, ...}
theory CommutativeGroup : FOL {include SemiGroup, ...}
theory Ring : FOL {
  additive: CommutativeGroup
  multiplicative: Semigroup
  ...
}
```

The UniFormal Library

Large Scale Example: The LATIN Atlas

- DFG project 2009–2012 (with DFKI Bremen and Jacobs Univ.)
- Highly modular network of little logic formalizations
  - separate theory for each
    - connective/quantifier
    - type operator
    - controversial axioms e.g., excluded middle, choice, ...
    - base type
  - reference catalog of standardized logics
  - documentation platform
- Written in MMT/LF
- 4 years, with $\sim$ 10 students, $\sim$ 1000 modules

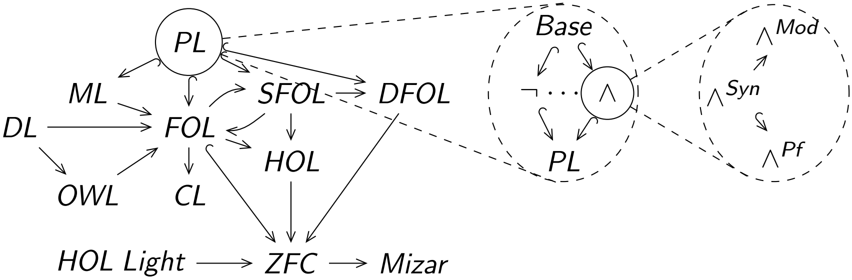
The LATIN Atlas of Logical Systems

It's big — that's me pointing at first-order logic

Logic Diagrams in LATIN

An example fragment of the LATIN logic diagram

- nodes: MMT/LF theories
- edges: MMT/LF theory morphisms

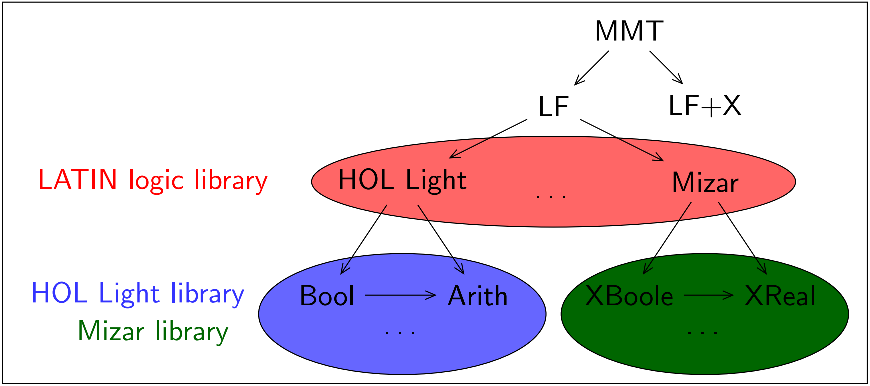


- each node is root for library of that logic
- each edge yields library translation functor

library *integration* very difficult though

OAF: Integration of Proof Assistant Libraries

- DFG project, 2014–2020, 15 contributors
- Big, overlapping libraries joined in MMT as the uniform representation language $>$ 100 GB XML in total
  Mizar, HOL systems, IMPS, Coq, PVS, Isabelle...
- enables archival, comparison, integration



OpenDreamKit: Virtual Math Research Environments

- EU project, 2015-2019, 15 sites, 25 partners
  http://opendreamkit.org/
- MMT as mediator system
  - system-independent formalization of math > 200 theories
    no proofs, no algorithms
  - integration of math computation systems
    SageMath, GAP, Singular: services interfaces defined in MMT
  - ...and math databases
    LMFDB, OEIS: database schemas defined in MMT

Example: dynamic retrieval

- SageMath user needs 13th transitive group with conductor 5
- SageMath queries MMT
- MMT retrieves it from LMFDB, translates it to SageMath syntax

OAF Overview

GitHub-like but for MMT projects https://gl.mathhub.info

- 251 Repositories
- 187 Users

For example:

Language	Library	Modules	Declarations
MMT	Math-in-the-Middle	220	826
LF	LATIN	529	2,824
PVS	Prelude+NASA	974	24,084
Isabelle	Distribution+AFP	12,318	2,116,638
HOL Light	Basic	189	22,830
Coq	$>$ 50 in total	1,979	167,797
Mizar	MML	1,194	69,710
SageMath	Distribution	1,399	
GAP	Library		9,050

OAF Example: The Isabelle Library

One of the most mature and widely used proof assistants

- 82 out of Wiedijk's top 100 math theorems formally proved
- L4 microkernel verification: $> 10^5$ loc
- Archive of Formal Proof
  > 300 authors, > 500 articles, $> 10^5$ lemmas, $> 10^6$ loc

Exported entire content to MMT — with M. Wenzel

- ≈ 9 person-months of work
- input
  - > 10k theories/locales, > 1M definitions and theorems
  - > 7000 files, 160 MB text (30 MB compressed)
- output (without proofs)
  - 65 GB XML (310 MB compressed)
  - 2M RDF individuals, 400M triples
- resource use: 8 cores, 80 GB RAM, 20h

Current Case Study: Universal Algebra

Algebraic hierarchy

- Easy and elegant to formalize in MMT
- But numerous generic operators

  homomorphisms, congruences, submodels, ...
- Tedious and error-prone to formalize individually

Diagram Operators

with Carette/Farmer/Sharoda

- Functors in the category of theories
- Natural transformations between them
- Preserve modular structure

  crucial to obtain human-readable theories
- Allow systematically generating large diagrams

Discussion

Foundation-Independent Development

Typical workflow

1. choose foundation
   type theories, set theories, first-order logics, higher-order logics, ...
2. implement kernel
3. build support algorithms — reconstruction, proving, editor, ...
4. build library

Foundation-independent workflow in MMT

1. MMT provides generic kernel
   no built-in bias towards any foundation
2. build support algorithms on top of MMT
3. choose foundation(s)
4. customize MMT kernel for foundation(s)
5. build foundation-spanning universal library

Advantages

- Avoids segregation into mutually incompatible systems
- Formulate maximally general results

  meta-theorems, algorithms, formalizations
- Rapid prototyping for logic systems

  customize MMT as needed, reuse everyting else
- Separation of concerns between
  - foundation developers
  - support service developers: search, axiom selection, ...
  - application developers: IDE, proof assistant, wiki, ...
- Allows evolving foundation along the way

  design flaws often apparent only much later
- Migrate formalizations when systems die
- Archive formalizations for future rediscovery

Paradigms

- judgments as types, proofs as terms
  unifies expressions and derivations
- higher-order abstract syntax — unifies operators and binders
- category of theories and theory morphisms
  - languages as theories
    unifies logical theories, logics, foundations
  - relations as theory morphisms
    unifies modularity, intepretations, representation theorems
- institution-style abstract model theory
  uniform abstract concepts
- models as morphisms (categorical logic)
  unifies models and translations and semantic interpretations

MMT Tool

Mature implementation

- API for representation language — foundation-independent
- Collection of reusable algorithms — no commitment to particular application
- Extensible wherever reasonable — storage backends, file formats, user interfaces, . . . operators and rules, language features, checkers, . . .

Separation of concerns between

- Foundation developers — e.g., language primitives, rules
- Service developers — e.g., search, theorem prover
- Application developers — e.g., IDE, proof assistant

Yields rapid prototyping for logic systems

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Yields rapid prototyping for logic systems

But how much can really be done foundation-independently?
MMT shows: not everything, but a lot

OpenDreamKit

Virtual Research Environments for Mathematics

- OpenDreamKit project 2015-2019 open PhD positions!
  EU project, 15 sites, 25 partners
  `http://opendreamkit.org/`
- Support full life-cycle
  - exploration, development, and publication
  - archival and sharing of data and computation
  - real mathematicians as target audience
- Key requirements
  - allow using any foundation
    any programming language, database
  - integrate narration, computation, databases
  - uniform user interfaces

Math in the Middle Approach

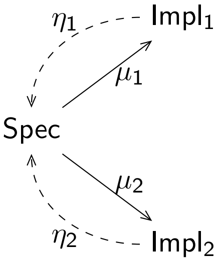
- Official definitions represented narratively
  MMT embedded into LaTeX to attach types
- Foundations written as MMT formal theories
- Computation: library interfaces exported as MMT theories
- Database schemata: written as formal MMT theories

Example work flow:

1. user input mentions specific object
   e.g., the 13th transitive group with conductor 5
2. Systems X queries MMT for $13a5$
3. MMT retrieves object from connected databases
   e.g., LMFDb (L-functions and modular forms)
4. Database schema defines type and encoding of object
5. MMT builds $13a5$ in high-level foundation
6. MMT exports $13a5$ in input syntax of system X

Library Integration

- Spec: mathematical theory axiomatic, foundation-independent
- Impl_i: implementation of Spec in system i
- μ_i: theory morphism describing how Spec is realized
  maps Spec-identifiers to their implementing objects
- η_i: partial inverse of μ_i



Challenge:

- collect initial library of mathematical concepts
- collect alignments with individual libraries

Mathematical Data Integration

Challenges of mathematical datasets

- tables using complex mathematical finite graphs, elliptic curves, ...
- possibly huge number of rows e.g., enumeration of all finite groups
- possibly large entries e.g., large integers, polynomials

requires optimized and math-aware database

Schema Theories

- MMT theory to represent relational table one constant per column
- MMT theory morphisms (= models) represent rows
  one interpretation per constant

Standardized Codecs

- Represent math types as SQL types
  e.g., integers as lists of digits in base 2^{64}
- Annotate constants in schema theory with codec
- MMT generates SQL tables and integration with database

MMT-Based Foundation-Independent Results

Logical Result: Representation Language

- MMT theories uniformly represent
  - logics, set theories, type theories, algebraic theories, ontologies, ...
  - module system: state every result in smallest possible theory

    Bourbaki style applied to logic
- MMT theory morphisms uniformly represent
  - extension and inheritance
  - semantics and models
  - logic translations
- MMT objects uniformly represent
  - functions/predicates, axioms/theorems, inference rules, ...
  - expressions, types, formulas, proofs, ...
- Reuse principle: theorems preserved along morphisms

Logical Result: Concepts

MMT allows coherent formal definitions of essential concepts

- Logics are MMT theories
- Foundations are MMT theories — e.g., ZFC set theory
- Semantics is an MMT theory morphism — e.g., from FOL to ZFC
- Logic translations are MMT theory morphisms
- Logic combinations are MMT colimits

Logical Results: Algorithms

- Module system
  modularity transparent to foundation developer
- Concrete/abstract syntax
  notation-based parsing/presentation
- Interpreted symbols, literals
  external model/implementation reflected into MMT
- Type reconstruction
  foundation plugin supplies only core rules
- Simplification
  rule-based, integrated with type reconstruction

- Theorem proving?
- Code generation? Computation?

Modular Framework Definitions in MMT

Individual features given by set of symbols, notations, rules

- $\lambda\Pi$
- Rewriting
- Polymorphism
- Subtyping (ongoing)
- ...

Language definitions are modular themselves

e.g., Dedukti = LF + rewriting

Knowledge Management Results

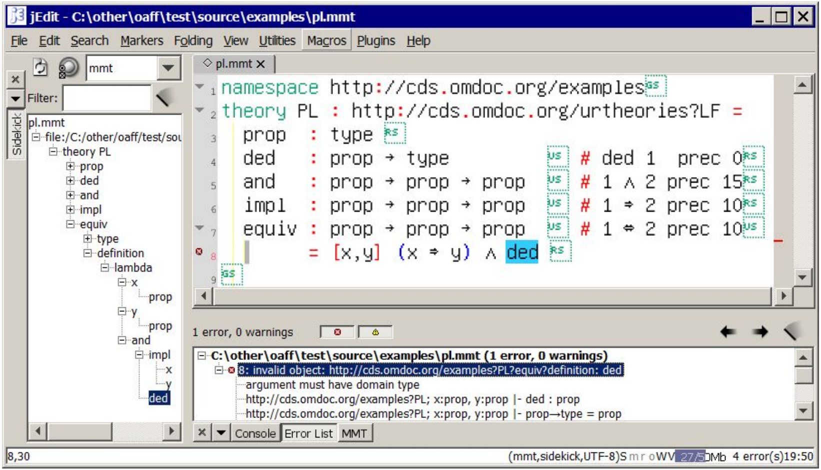
- Change management — recheck only if affected
- Project management — indexing, building
- Extensible export infrastructure — Scala, SVG graphs, LaTeX, HTML, . . .
- Search, querying — substitution-tree and relational index
- Browser — interactive web browser
- Editing — IDE-like graphical interface

IDE

- Inspired by programming language IDEs
- Components
  - jEdit text editor (in Java): graphical interface
  - MMT API (in Scala)
  - jEdit plugin to tie them together

    only $\sim$ 1000 lines of glue code
- Features
  - outline view
  - error list
  - display of inferred information
  - type inference of subterms
  - hyperlinks: jump to definition
  - search interface
  - context-sensitive auto-completion: show identifiers that

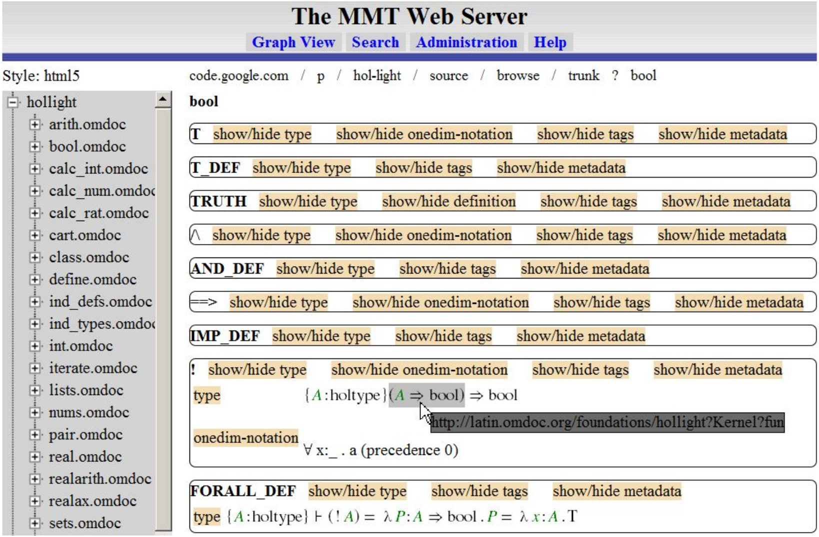
IDE: Example View



An Interactive Library Browser

- MMT content presented as HTML5+MathML pages
- Dynamic page updates via Ajax
- MMT used through HTTP interface with JavaScript wrapper
- Features
  - interactive display — e.g., inferred types, redundant brackets
  - smart navigation via MMT ontology — can be synchronized with jEdit
  - dynamic computation of content — e.g., definition lookup, type inference
  - graph view: theory diagram as SVG

Browser: Example View

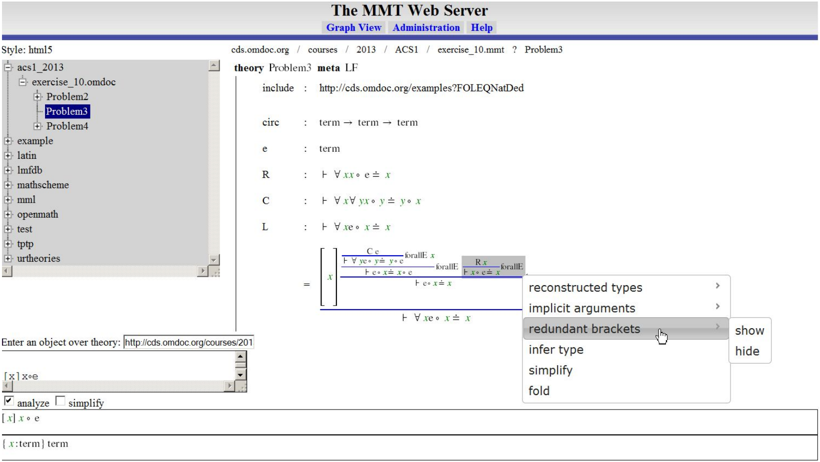


Browser Features: 2-dimensional Notations

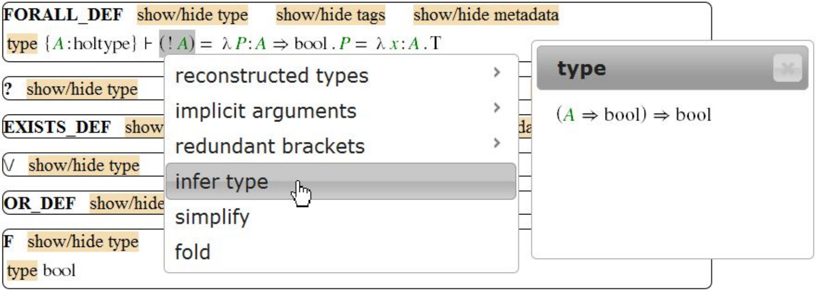
REAL_POW_DIV show/hide type show/hide definition

type $\vdash \forall x{:}\mathrm{real}\,.\,\forall y{:}\mathrm{real}\,.\,\forall n{:}\mathrm{num}\,.\left(\frac{x}{y}\right)^{n} = \frac{x^{n}}{y^{n}}$

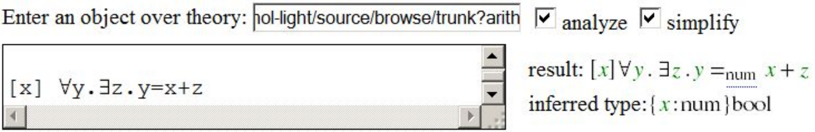
Browser Features: Proof Trees



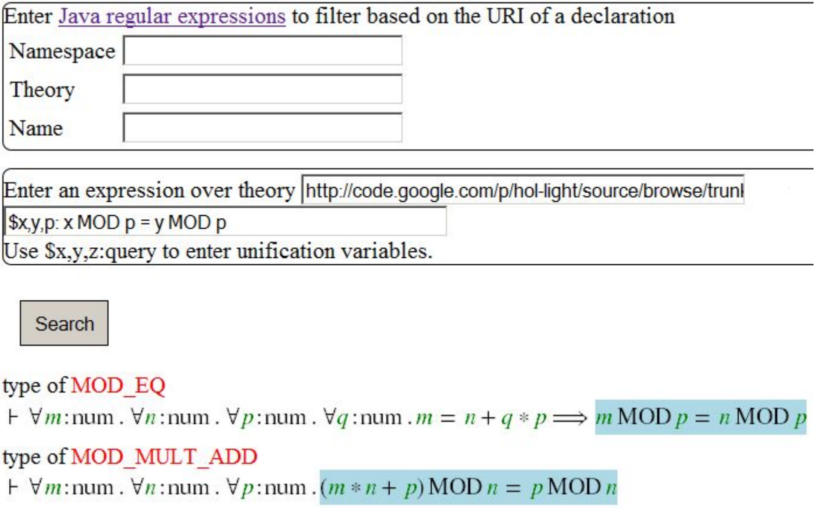
Browser Features: Type Inferece



Browser Features: Parsing



Example Service: Search



Example Service: Theory Graph Viewer

Theory graphs with 1000s of nodes

→ special visualization tools needed recently even in 3D

demo at `https://www.youtube.com/watch?v=Mx7HSWD5dwg`

Envisioned: A Generic Theorem Prover

- Theorem proving currently highly logic-specific
- But many successful designs actually logic-independent, e.g.,
  - Declarative proof languages
  - Tactic languages
  - Integration of decision procedures
  - Axiom selection
  - Modularity

Claim

- Logic-specific implementations a historical accident
- Possible to build MMT level proving technology

Envisioned: Computational Knowledge

- So far: MMT focuses on declarative formal languages
- Goal: extend to programming languages
  - understand key concepts foundation-independently
    state/effects, recursion, inheritance, ...
  - represent features modularly
  - freely mix logic and computation   share declarative aspects
- Syntax is (kind of) easy:
  - programming languages are MMT theories
  - classes/modules are MMT theories
  - programs are expressions
- Semantics is open question: What is the general judgment?

Big Challenges

Exporting System Libraries

Major interoperability obstacle

- Communities very focused on
  - honing their system
  - curating their library
- Little effort for making libraries accessible
- Slowly improving
  - more people interested in sharing and generic tools
  - OAF a driving force

MMT good candidate for truly universal standard

Universal Library of Elementary Knowledge

Requirements

- language-independent
- system-independent
- easy to write for practitioners

Vision

- Central library uses modular library of language features
- Focus on names, types, properties   no definitions, no proofs
- Individual contributions piece together minimal needed language
- All existing systems
  - remain in use   e.g., for optimized proof support
  - allow refactoring results for submission to central library

Summary

- MMT: foundation-independent framework for declarative languages
  - representation language
  - implementation
- Easy to instantiate with specific foundations

  rapid prototyping logic systems
- Deep foundation-independent results
  - logical: parsing, type reconstruction, module system, ...
  - knowledge management: search, browser, IDE, ...
- MMT quite mature now, ready for larger applications

  about to break even
- Serious contender for
  - universal library
  - generic applications/services
  - system integration/combination

Details

Further Resources

Web sites

- MMT: http://uniformal.github.io
- OAF: http://gl.mathhub.info/

Selected publications all available from https://kwarc.info/people/frabe

- the original paper on the MMT language (I&C 2013, with M. Kohlhase):
  A Scalable Module System
- a more recent paper on the MMT approach to logic (JLC 2014):
  How to Identify, Translate, and Combine Logics?
- a high-level paper on the goals of MMT (Log Univ 2015):
  The Future of Logic: Foundation-Independence
- the main technical result of MMT type checker (ToCL 2018):
  A Modular Type Reconstruction Algorithm
- Foundations in LATIN: (MSCS 2011, with M. Iancu)
  Formalizing Foundations of Mathematics
- Modular logics in LATIN (TCS 2011, with F. Horozal):
  Representing Model Theory in a Type-Theoretical Logical Framework
- the motivation paper on OAF (JFR 2015, with M. Kohlhase)
  QED Reloaded
- the main results of OAF (JAR 2021, with M. Kohlhase)
  Experiences from Exporting Major Proof Assistant Libraries

Details: Foundations

Foundations

- Foundation = the most primitive formalism on which everything else is built

  set theories, type theories, logics, category theory, ...

- We can fix the foundation once and for all — but which one?

- In math: usually implicit and arbitrary foundation
  - can be seen as avoiding subtle questions
  - but also as a strength: it's more general
- In CS: each system fixes its own foundational language

  e.g., a variant of type theory or HOL

- Programming languages foundations as well

  but representation of state in MMT still open problem

Fixed Foundations

- Fixing foundation the first step of most implementations
  often foundation and implementation have the same name
- No two implementations for the exact same foundation
  even reimplementations diverge quickly
- Negative effects
  - isolated, mutually incompatible systems
    no sharing of results, e.g., between proof assistants
  - no large scale libraries
    each system's library starts from scratch
  - no library archival
    libraries die with the system
  - comparison of systems difficult
    no common problem set
  - slow evolution
    evaluating a new idea can take years

Details: Logical Frameworks

Logical Frameworks

= meta-logic in which syntax and semantics of object logics are defined

Advantages

- Universal concepts — expressions, substitution, typing, equality, ...
- Meta-reasoning — consistency, logic translations, ...
- Rapid prototyping — type reconstruction, theorem proving, ...

Simplicity vs. expressivity

- Meta-logic must be simple to be scalable, trustworthy
- Object logic must be expressive to be practical
- Big challenge for frameworks

Designing Logical Frameworks

Typical = λ-calculus + X

Problems

- Divergence due to choice of features
  - logic programming (λ-Prolog)
  - meta logic (Twelf, Abella)
  - proof assistant for object logic (Isabelle)
  - reasoning about contexts (Beluga)
  - rewriting (Dedukti)
  - user-defined unification hints (ELPI)
- Even hypothetical union not expressive enough
  - can handle textbook logics — FOL, HOL, etc.
  - but not real-life logics — e.g., HOL Light, Mizar, PVS

Customizable Formal Systems

Parallel trend in formal system design

- increasingly complex problem domains
  e.g., mathematics, programming languages
- plain formalization introduces too many artifacts to be human-readable
- therefore: allow users to define how to interpret human input
  e.g., custom parsing, type reconstruction

Examples:

- unification hints (Coq, Matita)
  - extra-logical declarations
  - allow users to guide incomplete algorithms (e.g., unification)
- meta-programming (Idris, Lean)
  - expose internal datatypes to user
  - allow users to program extensions in the language itself

Details: MMT Syntax

Abstract Syntax of Terms

Key ideas

- no predefined constants
- single general syntax tree constructor $c(\Gamma; \vec{E})$
- $c(\Gamma; \vec{E})$ binds variables and takes arguments
  - non-binding operators: Γ empty   e.g., $\mathtt{apply}(\cdot; f, a)$ for $(f\ a)$
  - typical binders: Γ and $\vec{E}$ have length 1
    e.g., $\mathtt{lambda}(x\!:\!A; t)$ for $\lambda x\!:\!A.t$

contexts	Γ	$::=$	$(x[: E][= E])^*$
terms	E	$::=$	
constants		$\vert$	c
variables		$\vert$	x
complex terms		$\vert$	$c(\Gamma; E^*)$

Terms are relative to theory T that declares the constants c

Concrete Syntax of Terms

- Theories may attach notation(s) to each constant declaration
- Notations of c introduce concrete syntax for $c(\Gamma; \vec{E})$

e.g., for type theory

concrete syntax	constant declaration	abstract syntax
E ::=		
`type`	`type` #	`type`
$\Pi x : E_1.E_2$	`Pi` # Π V1 . 2	`Pi`$(x : E_1; E_2)$
$E_1 \rightarrow E_2$	`arrow` # 1 → 2	`arrow`$(\cdot; E_1, E_2)$
$\lambda x : E_1.E_2$	`lambda` # λ V1 . 2	`lambda`$(x : E_1; E_2)$
$E_1\ E_2$	`apply` # 1 2	`apply`$(\cdot; E_1, E_2)$

Notations

MMT implements parsing and rendering foundation-independently
relative to notations declared in current theory

Notations			$(ARG \mid VAR \mid DELIM)^*[PREC]$
Bound variable	VAR	$::=$	Vn for $n \in \mathbb{N}$
Argument	ARG	$::=$	n for $n \in \mathbb{N}$
Delimiter	$DELIM$	$::=$	Unicode string
Precedence	$PREC$	$::=$	integer

Bound variabes	`forall`	#	$\forall V1.2$
Mixfix	`setComprehension`	#	$\{V1 \in 2 \vert 3\}$
Argument sequences	`plus`	#	$1 + \ldots$
Variable sequences	`forall`	#	$\forall V1, \ldots 2$
Implicit arguments	`functionComposition`	#	$4 \circ 5$

Abstract Syntax of Theories

- Theories are named lists of declarations
- Theory names yield globally unique identifiers for all constants
- Module system: Previously defined theories can be included/instantiated

theory declaration			$T = \{Dec^*\}$
	Dec	$::=$	
constant declaration		$\mid$	$c[: E][= E][\#Notation]$
theory inclusion		$\mid$	$\mathtt{include}\ T$
theory instantiation		$\mid$	$\mathtt{structure}\ c : T\ \mathtt{where}\{Dec^*\}$

Flattening: Every theory is semantically equivalent to one without inclusions/instantiations   intuition: theories are named contexts

Details: Typing

Judgments

- MMT terms subsume terms of specific languages
- Type systems singles out the well-typed terms

For any theory Σ:

$\vdash \Sigma$	$T = \{\Sigma\}$ is a valid theory definition
$\vdash_T \Gamma$	Γ is a valid context
$\Gamma \vdash_T t : A$	t has type A
$\Gamma \vdash_T E = E'$	E and E' are equal
$\Gamma \vdash_T \_ : A$	A is inhabitable

Two kinds of rules:

- MMT defines some global rules once and for all

  foundation-independent rules

- declared in MMT theories, subject to scoping

  foundation-specific rules

  LF: $\sim$ 10 rules for LF, $\sim$ 10 lines of code each

Foundation-Independent Rules

- Lookup rules for atomic terms over a theory $T = \{\Sigma\}$

$$\frac{c : A \text{ in } \Sigma}{\vdash_T c : A} \qquad \frac{c = t \text{ in } \Sigma}{\vdash_T c = t}$$

- Equivalence and congruence rules for equality
- Rules for well-formed theories/contexts

$$\frac{}{\vdash \cdot} \qquad \frac{\vdash \Sigma \quad [\vdash_\Sigma \_ : A] \quad [\vdash_T t : A]}{\vdash \Sigma,\ c[: A][= t]}$$

Foundation-Specific Rules

Adding rules

- declared in theories by referencing Scala objects
- subject to module system
- Scala objects developed outside of MMT, loaded at run-time
- contain arbitrary code, e.g., error reporting, tracing, I/O

only needed to set up a logical framework — then declarative rules
or as fallback to tweak behavior

$\sim$ 10 kinds of rules, e.g.,

- simplification: $\Gamma \vdash_T E =?$
- equality checking: $\Gamma \vdash_T E = E'$?
- type inference: $\Gamma \vdash_T t :?$
- type checking: $\Gamma \vdash_T t : A$?
- proving: $\Gamma \vdash_T ? : A$

Foundation-Specific Rules (2)

8 rules to get dependent functions

- type inference for Π, λ, application
- checking+equality at Π-type
- β-conversion
- deconstructing unknown types, applying unknown functions

straightforward to write
module system, type reconstruction transparent to rules

Other examples

- proof irrelevance
- generated rewrite rules
- algebraic simplification
- ...

Type Reconstruction

Type checking:

- input: judgement, e.g., $\Gamma \vdash_T t : A$
- output: true/false, error information

Type reconstruction

- input judgment with unknown meta-variables
  - implicit arguments, type parameters
  - omitted types of bound variables
- output: unique solution of meta-variables that makes judgement true
- much harder than type checking

MMT implements foundation-independent type reconstruction

- transparent to foundations
- no extra cost for foundation developer

MMT's Type Reconstruction Algorithm

Algorithm

- MMT implements foundation-independent rules
- visible foundation-specific rules collected from current context
- algorithm delegates to foundation-specific rules as needed

General algorithm takes care of

- unknown meta-variables
- delaying constraints
- definition expansion
- module system

transparent to foundation-specific rules

Details: Theory Morphisms

One More Basic Concept: Theory Morphisms

Theories

- uniform representation of
  - foundations — e.g., logical frameworks, set theories, ...
  - logics, type theories
  - domain theories — e.g., algebra, arithmetic, ...
- little theories: state every result in smallest possible theory — maximizes reuse

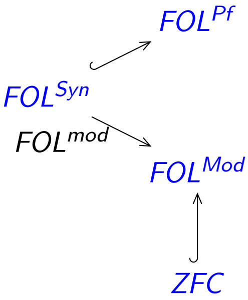
Theory morphisms

- uniform representation of
  - extension — e.g., Monoid → Group
  - inheritance — e.g., superclass → subclass
  - semantics — e.g., FOL → ZFC
  - models — e.g., Nat: Monoid → ZFC
  - translation — e.g., typed to untyped FOL
- homomorphic translation of expressions
- preserve typing (and thus truth)

Details: LATIN

Logic Example

- FOL^{Syn}: $term : type$, $prop : type$, $\mathtt{ded} : o \to type$, $\neg$, $\wedge$, ...
- FOL^{Pf}: $\neg I$, $\neg E$, $\wedge E_l$, $\wedge E_r$, $\wedge I$, ...
- ZFC: $set : type$, $wff : type$, $thm : wff \to type$, $\varnothing : set$, ...
- FOL^{Mod}: $univ : set$, $nonempty : true\ (univ \neq \varnothing)$
- FOL^{mod}: $term := univ$, $prop := \{0, 1\}$, $\mathtt{ded} := \lambda p.p \doteq 1$

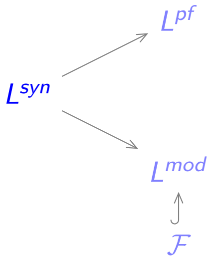


Current State

- Little theories including
  - propositional, common, modal, description, linear logic, unsorted/sorted/dependently-sorted first-order logic, CASL, higher-order logic
  - λ-calculi (λ-cube), product types, union types, ...
  - ZFC set theory, Mizar's set theory, Isabelle/HOL
  - category theory
- Little morphisms including
  - relativization of quantifiers from sorted first-order, modal, and description logics to unsorted first-order logic
  - negative translation from classical to intuitionistic logic
  - translation from type theory to set theory
  - translations between ZFC, Mizar, Isabelle/HOL
  - Curry-Howard correspondence between logic, type theory, and category theory

Representing Logics in LATIN

- L^{syn}: Syntax of L: connectives, quantifiers, etc.
  e.g., $\Rightarrow: o \to o \to o$
- L^{pf}: Proof theory of L: judgments, proof rules
  e.g., $impE \; : \; \mathtt{ded}\,(A \Rightarrow B) \to \mathtt{ded}\,A \to \mathtt{ded}\,B$
- L^{mod}: Model theory of L in terms of foundation $\mathcal{F}$
  e.g., $univ : set$, $nonempty : true\,(univ \neq \emptyset)$



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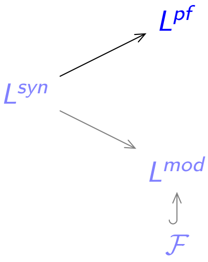

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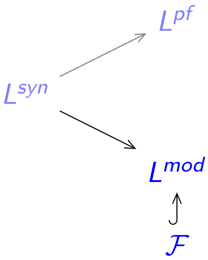
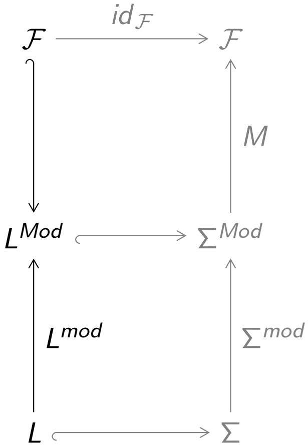


Representing Logics and Models

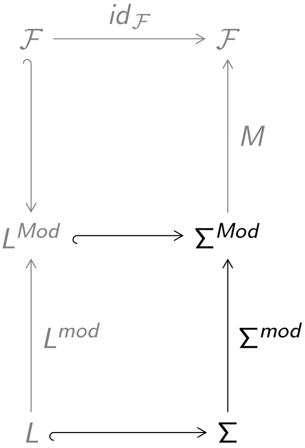


L encodes syntax and proof theory
$\mathcal{F}$ encodes foundation of mathematics
L^{Mod} axiomatizes models
L^{mod} interprets syntax in model

Σ encodes a theory of L,
extends L with functions, axioms, etc.
Σ^{Mod} correspondingly extends L^{Mod}
Σ^{mod} interprets syntax in model

M encodes a model of Σ,
interprets free symbols of L^{Mod} and Σ^{Mod}
in terms of $\mathcal{F}$

Representing Logics and Models



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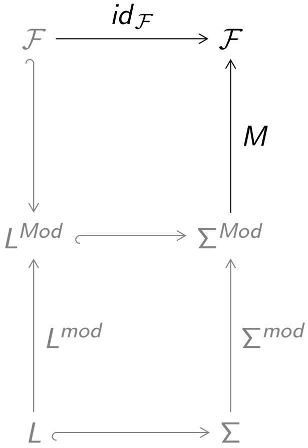

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Details: Open Archive of Formalizations

Goal: Universal Library Infrastructure

- MMT as representation language
- Repository backend: MathHub
  - based on GitLab – open-source analog of GitHub server
  - GitLab instance hosted at Jacobs University
  - free registration of accounts, creation of repositories
- Generic library management
  - browser
  - inter-library navigation
  - search
  - change management

Goal: Exports from Proof Assistants

- Export major libraries into MMT
- Representative initial targets
  - Mizar: set theoretical initial export done (with Josef Urban)
  - HOL Light: higher-order logic initial export done (with Cezary Kaliszyk)
  - Coq or Matita: type theoretical
  - IMPS: little theories method
  - PVS: rich foundational language
- Major technical difficulty
  - exports must be written as part of proof assistant
  - not all information available

Goal: Towards Library Integration

- Refactor exports to introduce modularity
- 2 options
  - systematically during export
    e.g., one theory for every HOL type definition
  - heuristic or interactive MMT-based refactoring
- Collect correspondences between concepts in different libraries
  heuristically or interactively
- Relate isomorphic theories across languages
- Use partial morphisms to translate libraries

Details: Systems

MMT and Hets

- Hets
  - integration system for specification languages and associated tools
  - developed at DFKI by Till Mossakowski et al.
  - limited foundation-independence
    - multiple foundations possible
    - but must be implemented individually
- Integration
  1. logics defined declaratively in MMT
  2. MMT generates Hets logic definitions

     includes logic-specific abstract data types
  3. new logics dynamically available in Hets
- Generated logics use MMT via HTTP for parsing/type-checking

LaTeX Integration

- MMT declarations spliced into LaTeX documents
  shared MMT-LaTeX knowledge space
- LaTeX macros for MMT-HTTP interface
- Semantic processing of formulas
  - parsing
  - type checking
  - semantic enrichment: cross-references, tooltips
- Design not LaTeX-specific
  e.g., integration with word processors possible

LaTeX Integration: Example

Inferred arguments are inserted during compilation:

- upper part: LaTeX source for the item on associativity
- lower part: pdf after compiling with LaTeX-MMT
- type argument M of equality symbol is inferred and added by MMT

```
\begin{mmtscope}
 For all \mmtvar{x}{in M},\mmtvar{y}{in M},\mmtvar{z}{in M}
 it holds that !(x * y) * z = x * (y * z)!
\end{mmtscope}
```

A *monoid* is a tuple $(M, \circ, e)$ where

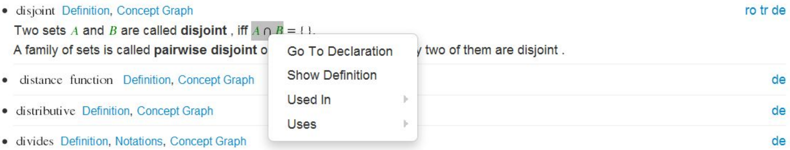
- M is a sort, called the universe.
- $\circ$ is a binary function on M.
- e is a distinguished element of M, the unit.

such that the following axioms hold:

- For all x,y,z it holds that $(x \circ y) \circ z =_M x \circ (y \circ z)$
- For all x it holds that $x \circ e =_M x$ and $e \circ x =_M x$.

SMGloM glossary

- Multi-lingual mathematical glossary
  https://gl.mathhub.info/smglom
  Kohlhase and others, 2013–2015
- Includes notations and verbalizations
  ... but makes no commitment to formal system
- Cross-referenced and searchable
- $\approx$ 1000 entries
- Uses MMT as background representation language
  integrates MMT with natural language



MMT in the GLF System

Uses MMT to combine

- Grammatical Framework GF for describing natural languages
- Logical Framework LF for describing logical languages
- Theorem provers to reason about logics

Semantics of natural language via
MMT-theory morphisms from GF-language to LF-language

Future: apply to semiformal language of mathematics
Fresh PhD student Jan Frederik Schaefer

MMT in the Semantic Alliance System

- Semantic Alliance System
  - developed by Michael Kohlhase et al.
  - two DFG projects
    - SiSSi (Hutter, Kohlhase) — spreadsheet applications
    - FormalCAD (Kohlhase, Schröder) — CAD applications
- Enrich domain-specific applications with semantic services
- Ontology used to
  - formalize background knowledge
  - share knowledge between applications
- MMT used as interface to ontology