TENSOR PRODUCTS, MULTIMAPS & INTERNAL HOMS

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INTRO

- 3 closely related concepts

Multicats

$\begin{matrix} a \\ b \end{matrix} \succ c$

Monoidal cats — Closed cats

$a \otimes b \to c$ $\qquad$ $a \to [b, c]$

- Exploring relationship between them leads to <u>skew</u> variants
- Tell this story
- How I got interested
- Examples
- Curious features

MONOIDAL CATS (Maclane, Benabou 63)

- Categorify monoids
- "Tensor product" $\otimes : \mathcal{C}^2 \to \mathcal{C}$

+ "unit" $I \in \mathcal{C}$

+ natural isos

$\alpha : (AB)C \cong A(BC)$, $l : IA \cong A$, $r : AI \cong A$

sat

5 axioms : $ABCD$ ← Maclane pentagon

IAB AIB ABI

II

- 3 axioms redundant (Kelly 64)
- These axioms generate "all" – Maclane's coherence theorem
- Eg. $(\mathrm{Vect}, \otimes)$, $(\mathrm{Set}, \times)$

CLOSED CATS (Eilenberg-Kelly '65)

- "Internal hom" $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$
- "unit" $I \in \mathcal{C}$
- extranatural families

$L : [A,B] \longrightarrow [[C,A],[C,B]]$,

$i : [I,A] \cong A$,

$j : I \longrightarrow [A,A]$ sat 5 axioms

$+\ \ \mathcal{C}(A,B) \longrightarrow \mathcal{C}(I,[A,B])$

$f \longmapsto I \xrightarrow{j} [A,A] \xrightarrow{f_*} [A,B]$

invertible

- As before, 3 transformations, 5 axioms, only 2 invertible maps

Vect, $[A,B] = \mathrm{Vect}(A,B)$ with pointwise str.

CLOSED CATS 2

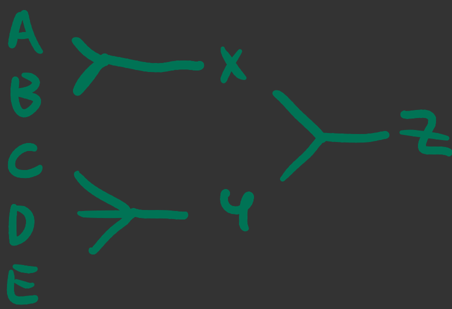
- First monograph on enriched cats used closed cats
- Often easy to describe internal homs explicitly, unlike tensor products!
- Axiomatics involve iterated contravariance & hard to read!

MULTICATS (Lambek 69)

$\mathbb{C}$:- objects A, B, C

- n-ary multimaps 

all $n \in \mathbb{N}$
- unary identities +
substitution

e.g. 

satisfying assoc. & unit laws.

- Vector spaces and multilinear maps
- Sequents $A_1, \dots, A_n \vdash B$ in sequent calculus
- Cat $\mathcal{C}$ of unary maps :
  i.e. cats with structure.

EILENBERG-KELLY THM

- Cat $\mathcal{C}$ + $I \in \mathcal{C}$ + $\otimes : \mathcal{C}^2 \to \mathcal{C}$ + $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ + nat isos $\mathcal{C}(AB, C) \cong \mathcal{C}(A, [B,C])$.
- Then there is bijection betw.

① Ext^n of $(\mathcal{C}, \otimes, I)$ to monoidal cat

② Ext^n of $(\mathcal{C}, [-,-], I)$ to closed cat such that certain map $t : [A \otimes B, C] \to [A, [B,C]]$ is invertible.

$\boxed{\text{Proof}}$ Perfect corr between

- transforms α, ℓ, r ~ L, j, i
- 5 axioms ~ 5 axioms
- 3 invertible maps ~ 2 invertible maps + t to fix it!

Imbalance : Nothing on closed side ~ invertibility of associator.

Rectifying imbalance - going skew!

- Let's <u>uninvert some isos</u> - easier on closed side.
- Not historically correct approach by lit.

Skew closed cats (Street 2013)

- "Internal hom" $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$
- "unit" $I \in \mathcal{C}$

extranat. families

$L : [A,B] \longrightarrow [[C,A],[C,B]]$,

$i : [I,A] \longrightarrow A$,

$j : I \longrightarrow [A,A]$ sat $\underline{5}$ axioms as before.

Skew monoidal cats (Szlachányi '12)

- "Tensor product" $\otimes : \mathcal{C}^2 \longrightarrow \mathcal{C}$
+ "unit" $I \in \mathcal{C}$
+ natural transformations

$\alpha : (AB)C \to A(BC)$,

$\ell : IA \to A$,

$r : A \to AI$

sat 5 axioms as before.

- Note orientation.
- No redundant axioms

Street's theorem

Cat $\mathcal{C}$ + $I \in \mathcal{C}$ +
$\otimes : \mathcal{C}^2 \to \mathcal{C}$ + $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$
+ nat isos $\mathcal{C}(AB, C) \cong \mathcal{C}(A, [B,C])$.

- Then there is bijection betw.

1. Ext^n of $(\mathcal{C}, \otimes, I)$ to skew mon cat
2. Ext^n of $(\mathcal{C}, [-,-], I)$ to skew closed cat.

- EK-theorem really a refinement of above
- Totality $(\mathcal{C}, \otimes, [-,-], I)$ as above called mon. skew closed or closed skew monoidal.
- Examples?

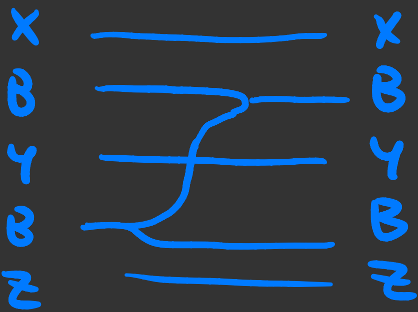
1. Szlachanyi's examples

- B a bialgebra in smc $\mathcal{C}$

- Skew mon str

$X * Y = XBY$, unit I

Assoc



units

- Can classify bialgebras & bialgebroids using such skew structures.

② Fancy theories as monoids

- $I : A \to \mathcal{E}$ functor & $\mathcal{E}$ admits left Kan extensions along I.
- Altenkirch-Uustalu show $(A, \mathcal{E})$ has skew mon. str.
- Unit is I
- T.product

$$\begin{tikzcd}
A \arrow[r, "G"'] \arrow[drr, bend left=60, "F * G"] & \mathcal{E} \arrow[dr, "\mathrm{lan}_I F"] & \\
& A \arrow[u, "I"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{E}
\end{tikzcd}$$

- Monoid in $(A, \mathcal{E})$ is relative monad.
- $(A, \mathcal{E})$ monoidal in good classical situations like $I := \mathbb{F} \hookrightarrow \mathrm{Set}$ but not others.
- Lack described operadic cats (generalising operads) as monoids in a skew mon. cat.

③ Weak maps & 2-cat theory (JB)

- In a skew closed cat,

$$\mathcal{C}(A,B) \xrightarrow{[A,-]\circ j} \mathcal{C}(I,[A,B])$$

maps $A \to B$ $\longmapsto$ "elements" of $[A,B]$ need not be invertible.

- Interpret this as:

in skew world, $[A,B]$ contains maps $A \to B$ but maybe other stuff too - eg. weak maps.

③ Weak maps & 2-cat theory 2

- Common situation in 2-cat theory
- E.g. $\mathrm{SMonCat}_s \hookrightarrow \mathrm{SMonCat}_p$

  symmetric mon cats, strict maps, s. mon transf — symmetric mon cats, strong maps, s. mon transf

- LHS better behaved — RHS of interest
- $\mathrm{SMonCat}_p$ has closed mon. bicat str – $[A,B]$ contains strong symm. mon. functors w/ pwise structure.
- Complex as mon bicat str is weak.
- Homs $[A,B]$ restrict to skew monoidal cl. str. on $\mathrm{SMonCat}_s$.
- Contains original monoidal bicat in a certain sense but better behaved categorically.
- Many such examples (JB 2016)

Curious features of skew structures

① $i\otimes -$ comonad w/ counit $ia \xrightarrow{\ell} a$
$-\otimes i$ monad w/ unit $a \xrightarrow{r} ai$
and comonad distributes over monad.

- Maps in Kleisli cat $IA \longrightarrow B$ for comonad can be thought of as weak maps.
- In 2-cat examples, comonad is pseudomorphism classifier

② (DUALITY!)

$\mathcal{C}$ skew monoidal $\Rightarrow$ $\mathcal{C}^{op}$ skew monoidal $A, B \longmapsto B\otimes A$.

③ ①,② suggest skew monoidal categories not lax or oplax structures but bilax, in some sense!

E.g. lax monoidal cats fail duality.

4. Skew monoidal cat does have lax monoidal str – not totally obvious:

$(a_1, \ldots, a_n) \longmapsto ia_1 \ldots a_n$ left bracketed with unit on left

& oplax monoidal str.

$(a_1, \ldots, a_n) \longmapsto a_1 \ldots a_n i$ right bracketed.

Remark: Many analogies with concept of algebraic weak factorisation system.

Skew multicats (Bourke-Lack)

- Monoidal cats $\overset{\text{(Hermida)}}{\sim}$ Representable multicats
- Closed cats $\overset{\text{(Manzyuk)}}{\sim}$ closed multicats w/ unit
- What is a skew multicat?
  - A multicat $\mathbb{C}_l$ of "loose multimaps"
  - For $n>0$ a function

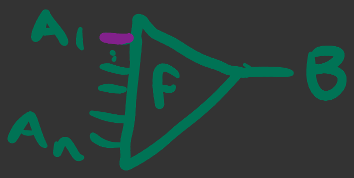
$$j_n : \underbrace{\mathbb{C}_t(A_1,\ldots,A_n;B)}_{\text{"tight multimaps"}} \longrightarrow \underbrace{\mathbb{C}_l(A_1,\ldots,A_n;B)}_{\text{"loose multimaps"}}$$

viewing tight multimaps as loose + extra sub. & conditions.

- In case, the j_n are inclusions (tights are special loose maps) these say:
  - identities are tight
  - $f\circ(g_1,\ldots,g_n)$ is tight if f & g_1 are.

Skew multicats ctd.

- Think tight multimap is loose one with special behaviour in 1'st variable



closed under substit.

- Eg. multimaps strict in 1'st variable
- 2-cat examples & many others arise in this way.
- Skew mon cats $\overset{\text{Bourke-}}{\underset{\text{Lack}}{\sim}}$ Left rep. skew multicats.
- Ongoing work connected w' this w G. Lobbia.
- Uustalu-Zeilberger & others studying connections between skew structures & sequent calculi (inspired by Lambek)

Thanks for listening !