A Topos View of Axioms of Choice for Finite Sets

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Introduction

Alfred Tarski proved, in set theory without the axiom of choice:
If every family of 2-element set has a choice function,
then so does every family of 4-element sets.

Proof: Given a family $\mathcal{F}$ of 4-element sets, choose an element from each of their 2-element subsets. So, for each $A \in \mathcal{F}$, you've made 6 choices. Not all 4 elements of A are chosen equally often. Consider those chosen most often. If there's only one, make it your choice from A. If there are 3, make the other one your choice. If there are 2, you've already chosen one from that pair.

Suppose every countable family of 2-element sets has a choice function. Does it follow that every countable family of 4-element sets has a choice function?
The preceding proof doesn't show that. And in fact it can be false.

Introduction, continued

In 1937, Andrzej Mostowski analyzed more general implications between axioms of choice for sets of specified, finite sizes.
He obtained number-theoretic and group-theoretic conditions for provability of such implications.
He had some necessary conditions and some sufficient conditions, but no necessary and sufficient conditions.

The first question not settled by his conditions was:
Assume that all families of 3-element sets, 5-element sets, and 13-element sets have choice functions.
Does it follow that all families of 15-element sets have choice functions?

In 1970, Robert Gauntt showed that one of Mostowski's group-theoretic sufficient conditions is also necessary.
In particular, he settled the 3,5,13,15-problem in the negative.

Notation and Terminology

$C(n, I)$ means every I-indexed family of n-element sets has a choice function.
$C(Z, I)$ means every I-indexed family of n-element sets for $n \in Z$ has a choice function.

Theorem (Tarski)

If $\forall I\, C(2, I)$ then also $\forall I\, C(4, I)$. *Global*

It is not claimed that

$$\forall I\, (C(2, I) \implies C(4, I)).$$

Theorem

$\forall I\, (C(\{2, 3\}, I) \implies C(4, I))$. *Local*

Proof that $C(\{2,3\}, I)$ implies $C(4, I)$

Given a family of 4-element sets, note that each has 3 partitions into two 2-element subsets. Choose one such partition for each of the given sets. Then choose one of the two pieces of that partition. Finally, choose an element from the chosen piece.

Mostowski-Gauntt Permutation Group Criteria

Theorem (Local)

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *ZFA proves* $\forall I\,(C(Z,I) \implies C(n,I))$.
- *Any group that can act without fixed points on n can also act without fixed points on some $z \in Z$.*

Theorem (Global)

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *ZFA proves "If* $\forall I\, C(Z,I)$ *then* $\forall I\, C(n,I)$."
- *Any group that can act without fixed points on n has a subgroup that can act without fixed points on some $z \in Z$.*

Topoi

A *topos* is a category $\mathcal{E}$ so similar to the category **Set** of sets and functions that it makes sense to speak of structures in $\mathcal{E}$, and to interpret first-order and even higher-order formulas in such structures.

Higher-order *intuitionistic* logic is sound for these interpretations.

I'll use "topos" to mean "elementary topos with natural numbers object".

Important Examples

$\mathbf{Set}^I = \mathbf{Set}/I$: topos of I-indexed families of sets and of functions.

Any model M of ZF. (Can allow atoms, can omit replacement, can omit regularity. Also M/I for any object I of M.)

Fix a group G.
G-**Set**: topos of sets with (left) action of G and functions that commute with the actions.

More Examples

M-sets for any monoid M

Kripke structures for intuitionistic logic

Presheaves on any small category

Sheaves on any topological space (or any site)

Realizability topoi

Facts about $\mathbf{Set}^I$

Let n be a natural number. An object $X = (X_i)_{i \in I}$ in $\mathbf{Set}^I$ is an n-element set, in the sense of the internal logic of $\mathbf{Set}^I$, iff each X_i is an n-element set.

A *point* in X is a morphism from $1 = (1)_{i \in I}$ to X. So it amounts to a choice of one element from each X_i.

Similarly when **Set** is replaced by other models of ZF.

So $C(n, I)$ says that, in $\mathbf{Set}^I$, every n-element set (in the sense of internal logic) has a point.

Notation: Abbreviate "In topos $\mathcal{E}$, every n-element set has a point" as $EP(n, \mathcal{E})$. And $EP(Z, \mathcal{E})$ means $EP(z, \mathcal{E})$ for all z in the finite set Z.

Facts about G-**Set**

A G-set X is an n-element set in the topos G-**Set** iff X has n elements (regardless of how G acts on it).

A point in an object X of G-**Set** is an element X fixed by the action of all $g \in G$.

So $EP(n, G\text{-}\mathbf{Set})$ says that every action of G on n has a fixed point. And $EP(Z, G\text{-}\mathbf{Set})$ says the same for actions of G on any $z \in Z$.

Local Mostowski-Gauntt in Topos Language

Theorem

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *For every topos $\mathcal{E}$ of the form* **Model of ZF**$/I$, $EP(Z, \mathcal{E})$ *implies* $EP(n, \mathcal{E})$.
- *For every topos $\mathcal{E}$ of the form* G-**Set**, $EP(Z, \mathcal{E})$ *implies* $EP(n, \mathcal{E})$.

What other sorts of topoi does this equivalence apply to?

General Local Theorem

Theorem

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *For every topos $\mathcal{E}$ of the form* **Model of ZF**$/I$, $EP(Z, \mathcal{E})$ *implies* $EP(n, \mathcal{E})$.
- *For every topos $\mathcal{E}$ of the form G-***Set**, $EP(Z, \mathcal{E})$ *implies* $EP(n, \mathcal{E})$.
- *For every topos $\mathcal{E}$,* $EP(Z, \mathcal{E})$ *implies* $EP(n, \mathcal{E})$.

General Global Theorem

Theorem

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *ZF (or ZFA) proves that, if $C(Z, I)$ for all I then $C(n, I)$ for all I.*
- *For every topos $\mathcal{E}$ of the form* **Model of ZF**$/I$*, if $EP(Z, \mathcal{E}/F)$ for all finite decidable objects F of $\mathcal{E}$, then $EP(n, \mathcal{E})$.*
- *Every group that acts without fixed points on n has a subgroup that acts without fixed points on some $z \in Z$.*
- *For every topos $\mathcal{E}$ of the form G-**Set**, if $EP(Z, \mathcal{E}/F)$ for all finite decidable objects F of $\mathcal{E}$, then $EP(Z, \mathcal{E})$.*
- *For every topos $\mathcal{E}$, if $EP(Z, \mathcal{E}/F)$ for all finite decidable objects F of $\mathcal{E}$, then $EP(n, \mathcal{E})$.*

Expected Extensions

There will be analogous results concerning choice principles like "Given a family of 7-element sets with specified orientations, there is a function assigning to each of these sets a 3-element subset."

The group-theoretic conditions will be about groups that can act on a 7-element set, preserving an orientation but not preserving any 3-element subset.

The topos-theoretic statements will be about objects that (in the sense of the internal logic) have 7 elements and an orientation, and ask whether the (internally defined) object of 3-element subobjects has a point.