Doubly Lax Colimits of Double Categories with Applications

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Double Categories

- A **double category** is an internal category in **Cat**,

$$\mathbf{C}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbf{C}_0 \ .$$

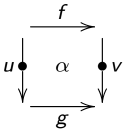
- It has
  - objects (objects of $\mathbf{C}_0$),
  - vertical arrows (arrows of $\mathbf{C}_0$), denoted $d_0(v) \overset{v}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} d_1(v)$,
  - horizontal arrows (objects of $\mathbf{C}_1$), denoted $s(f) \xrightarrow{f} t(f)$,
  - double cells (arrows of $\mathbf{C}_1$), denoted

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow v \\ A' & \xrightarrow[f']{} & B' \end{array}$$

where $d_0(\alpha) = f$, $d_1(\alpha) = f'$, $s(\alpha) = u$, and $t(\alpha) = v$.

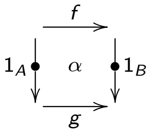
Examples

1. For any 2-category $\mathcal{C}$, $\mathbb{Q}(\mathcal{C})$ is the double category of quintets in $\mathcal{C}$, with double cells



for each $\alpha\colon vf \Rightarrow gu$ in $\mathcal{C}$.

2. For any 2-category $\mathcal{C}$, $\mathbb{H}(\mathcal{C})$ is the double category with double cells



for each $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$.

3. The double category $\mathbb{V}(\mathcal{C})$ is defined analogously.

The category **DblCat**

The category **DblCat** of double categories has:

- objects: double categories $\mathbb{C}, \mathbb{D}, \ldots$;
- arrows: double functors $F, G, \ldots$;
- 2-cells: these come in two flavours:
  - vertical transformations $\gamma \colon F \Longrightarrow\!\!\!\!\bullet\;\, G \colon \mathbb{C} \rightrightarrows \mathbb{D}$ given by

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ {\scriptstyle \gamma_A}\downarrow\!\bullet & \gamma_h & \bullet\!\downarrow{\scriptstyle \gamma_B} \\ GA & \xrightarrow[Gh]{} & GB \end{array} \quad \text{for each } h \colon A \to B \text{ in } \mathbb{C}$$

    functorial in the horizontal direction and natural in the vertical direction.
  - horizontal transformations $\nu \colon F \Longrightarrow G$ are defined dually;
- modifications given by a family of double cells.

The category **DblCat** - Properties

- **DblCat** is not a double category.
- **DblCat** is enriched in the category DblCat of double categories: each **DblCat**$(\mathbb{C}, \mathbb{D})$ is a double category.
- **DblCat**$_v$ (resp. **DblCat**$_h$) is the 2-category with vertical (resp. horizontal) transformations.
- So lax limits have typically been taken in the 2-category **DblCat**$_v$ or **DblCat**$_h$ with laxity in one direction.

Diagrams in **DblCat**

To define a diagram of double categories indexed by a double category $\mathbb{D}$:

- Send objects of $\mathbb{D}$ to double categories;
- Send both horizontal and vertical arrows to double functors;
- For 2-dimensional cells we have to make a choice: we send double cells to *vertical* transformations.

So an indexing double functor is a double functor

$$\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v)$$

We will also refer to indexing double functors as **vertical double functors**

$$\mathbb{D} \xrightarrow{v} \mathbf{DblCat}.$$

Questions

- Have we lost our ability to use horizontal transformations and modifications?
- Have we lost our ability to distinguish between horizontal and vertical arrows in the indexing double category?

No, they will show up in the notion of **doubly lax transformation**.

Intro to Doubly Lax Transformations

- Introduce a **cylinder double category** $\mathrm{Cyl}_v(\mathbf{DblCat})$.
- There are vertical double functors

$$\mathrm{Cyl}_v(\mathbf{DblCat}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbf{DblCat}$$

- A **doubly lax transformation** $\alpha\colon F \Rightarrow G\colon \mathbb{D} \to \mathbf{DblCat}$ is given by a double functor

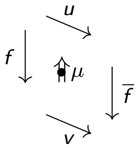
$$\alpha\colon \mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat})$$

such that $d_0\alpha = F$ and $d_1\alpha = G$.

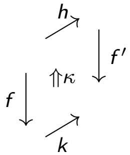
The Double Category of (Vertical) Cylinders

The double category $\mathrm{Cyl}_v(\mathbf{DblCat})$ of **vertical cylinders** is defined by:

- Objects are double functors, denoted by $\downarrow f$.
- Vertical arrows $f \xrightarrow{(u,\mu,v)} \overline{f}$ are given by vertical transformations,

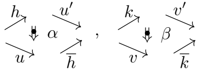


- Horizontal arrows $f \xrightarrow{(h,\kappa,k)} f'$ are given by horizontal transformations,



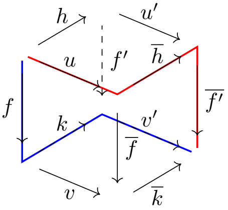
Double Cylinders

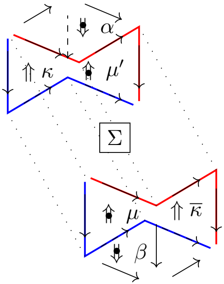
A double cell, $\begin{array}{ccc} f & \xrightarrow{(h,\kappa,k)} & f' \\ {\scriptstyle (u,\mu,v)}\Downarrow & (\alpha,\Sigma,\beta) & \Downarrow{\scriptstyle (u',\mu',v')} \\ \overline{f} & \xrightarrow[(\overline{h},\overline{\kappa},\overline{k})]{} & \overline{f'} \end{array}$ consists of two vertical 2-cells,



and a modification Σ,

$$\begin{array}{ccc} v'kf & \xRightarrow{v'\kappa} & v'f'h \\ \Downarrow{\scriptstyle \beta f} & & \Downarrow{\scriptstyle \mu' h} \\ \overline{k}vf & \Sigma & \overline{f'}u'h \\ \Downarrow{\scriptstyle \overline{k}\mu} & & \Downarrow{\scriptstyle \overline{f'}\alpha} \\ \overline{k}\,\overline{f}u & \xRightarrow{\overline{\kappa}u} & \overline{f'}\,\overline{h}u \end{array}$$





Cylinders and Transformations

- There are vertical double functors $d_0, d_1 \colon \mathrm{Cyl}_v(\mathbf{DblCat}) \xrightarrow{v} \mathbf{DblCat}$, sending a cylinder to its top and bottom respectively;
- A **doubly lax transformation** $\theta \colon F \Rightarrow G$ between vertical double functors $F, G \colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ is given by a double functor

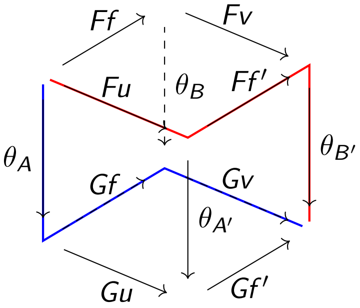
$$\theta \colon \mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat}),$$

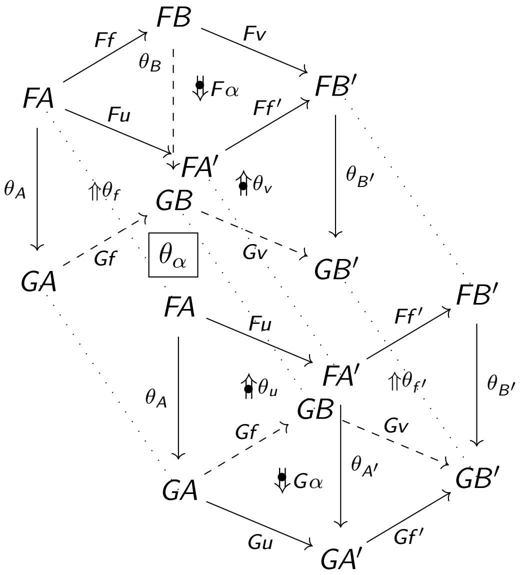
such that $d_0\theta = F$ and $d_1\theta = G$.

Doubly Lax Transformations $\theta\colon F \Rightarrow G$

For each double cell $\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow v \\ A' & \xrightarrow[f']{} & B' \end{array}$,

$$\begin{array}{ccc} GvGf\theta_A & \xRightarrow{Gv\theta_f} & Gv\theta_B Ff \\ G\alpha\theta_A \Downarrow\bullet & & \bullet\Downarrow \theta_v Ff \\ Gf'Gu\theta_A & \theta_\alpha & \theta_{B'}FvFf \\ Gf'\theta_u \Downarrow\bullet & & \bullet\Downarrow \theta_{B'}F\alpha \\ Gf'\theta_{A'}Fu & \xRightarrow[\theta_{f'}Fu]{} & \theta_{B'}Ff'Fu \end{array}$$

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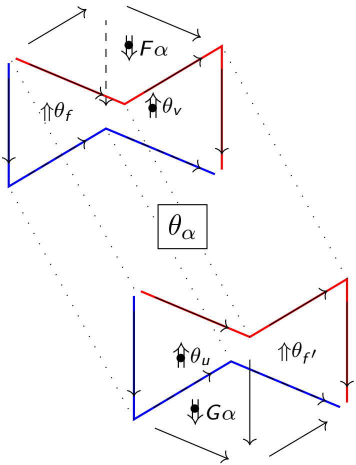
$F\alpha$
θ_f
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Doubly Lax Transformations

- Let $F, G \colon \mathbb{D} \xrightarrow{v} \mathbf{DblCat}$ be vertical double functors.
- Since doubly lax transformations $F \Rightarrow G$ are represented by double functors,

$$\mathbb{D} \to \mathrm{Cyl}_v(\mathbf{DblCat})$$

  they are the objects of a double category $\mathbb{H}\mathrm{om}_{d\ell}(F, G)$: a sub double category of $\mathbf{DblCat}(\mathbb{D}, \mathrm{Cyl}_v(\mathbf{DblCat}))$.

Lax Transformations Between 2-Functors

- By applying $\mathbb{Q}$ to the hom-categories of a 2-category $\mathcal{B}$, we can make it into a DblCat-enriched category $\widehat{\mathbf{Q}}(\mathcal{B})$.
- This allows us to view lax transformations between 2-functors as a special case of the new doubly lax transformations.

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \;\; \Downarrow \alpha \qquad \rightsquigarrow \qquad \mathbb{Q}\mathcal{A} \underset{\mathbb{Q}G}{\overset{\mathbb{Q}F}{\rightrightarrows}} \widehat{\mathbf{Q}}(\mathcal{B}) \;\; \Downarrow \alpha$$

- By taking a restricted $\mathbb{Q}$ on the codomain, taking only a particular class Ω of 2-cells of $\mathcal{B}$ for the local horizontal arrows, we obtain Ω-transformations.
- By taking a restricted $\mathbb{Q}$ on the domain, we also get Σ-transformations.

Doubly Lax Colimits

- A **doubly lax cocone** for a vertical double functor $F : \mathbb{D} \twoheadrightarrow \mathbf{DblCat}$ with vertex $\mathbb{E} \in \mathbf{DblCat}$ is a doubly lax transformation $F \overset{\theta}{\Rightarrow} \Delta\mathbb{E}$.
- There is a double category,

$$\mathbb{L}\mathsf{C}(F, \mathbb{E}) := \mathbb{H}\mathrm{om}_{d\ell}(F, \Delta\mathbb{E})$$

  of doubly lax cocones with vertex $\mathbb{E}$.
- A doubly lax cocone $F \overset{\lambda}{\Longrightarrow} \Delta\mathbb{L}$ is the **doubly lax colimit** of F if, for every $\mathbb{E} \in \mathbf{DblCat}$,

$$\mathbf{DblCat}(\mathbb{L}, \mathbb{E}) \xrightarrow{\lambda^*} \mathbb{L}\mathsf{C}(F, \mathbb{E})$$

  is an isomorphism of double categories.
- The doubly lax colimit can be obtained by a **double Grothendieck construction**.

The Double Grothendieck Construction: Objects and Arrows

Let $\mathbb{D} \xrightarrow[v]{F} \mathbf{DblCat}$ be a vertical double functor. The **double category of elements**, $\mathbb{G}\mathrm{r}\, F = \int_{\mathbb{D}} F$, is defined by:

- Objects: (C, x) with C in $\mathbb{D}$ and x in FC,
- Vertical arrows:
$$(C, x) \xrightarrow[\bullet]{(u,\rho)} (C', x'),$$
where $C \xrightarrow[\bullet]{u} C'$ in $\mathbb{D}$ and $Fux \xrightarrow[\bullet]{\rho} x'$ in FC'.
- Horizontal arrows:
$$(C, x) \xrightarrow{(f,\varphi)} (D, y),$$
where $C \xrightarrow{f} D$ in $\mathbb{D}$, and $Ffx \xrightarrow{\varphi} y$ in FD.

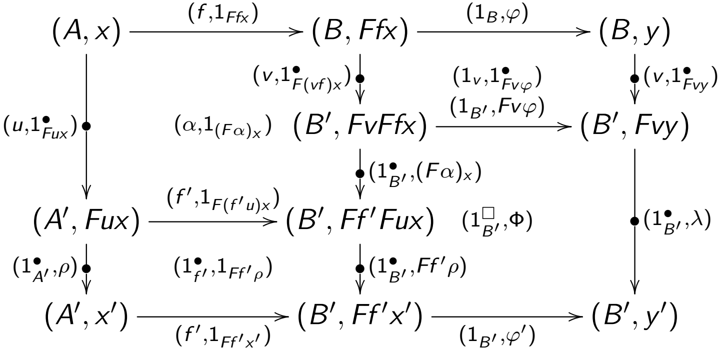
The Double Grothendieck Construction: Double Cells

- Double cells: $\begin{array}{ccc} (C,x) & \xrightarrow{(f,\varphi)} & (D,y) \\ {\scriptstyle (u,\rho)} \big\downarrow \bullet & (\alpha,\Phi) & \bullet \big\downarrow {\scriptstyle (v,\lambda)} \\ (C',x') & \xrightarrow[(f',\varphi')]{} & (D',y') \end{array}$, where $\alpha\colon \left(u \; {}^{f}_{f'} \; v \right)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FD':

$$\begin{array}{ccc} FvFfx & \xrightarrow{Fv\varphi} & Fvy \\ {\scriptstyle (F\alpha)_x} \big\downarrow \bullet & & \big| \\ Ff'Fux & \Phi & \bullet\, {\scriptstyle \lambda} \\ {\scriptstyle Ff'\rho} \big\downarrow \bullet & & \big\downarrow \\ Ff'x' & \xrightarrow[\varphi']{} & y' \end{array}$$

Factorization

- Any horizontal arrow (f, φ) can be factored as
  $(A, x) \xrightarrow{(f,1_{Ffx})} (B, Ffx) \xrightarrow{(1_B,\varphi)} (B, y)$.
- Any vertical arrow (u, ρ) can be factored as
  $(A, x) \xrightarrow{(u,1^\bullet_{Fux})} (A', Fux) \xrightarrow{(1^\bullet_{A'},\rho)} (A', x')$.
- And any double cell (α, Φ) can be factored as



The Main Theorem

- There is a doubly lax cocone $F \stackrel{\lambda}{\Longrightarrow} \Delta \mathbb{G}\mathrm{r}\, F$ with the required universal property:

$$\lambda^* \colon \mathbf{DblCat}\left(\int_{\mathbb{D}} F, \mathbb{E}\right) \to \mathbb{L}\mathrm{C}\left(\int_{\mathbb{D}} F, \mathbb{E}\right)$$

  is an iso of double categories for all $\mathbb{E} \in \mathbf{DblCat}$.

- Furthermore, $\int_{\mathbb{D}}$ extends to a functor of DblCat-categories

$$\mathrm{Hom}_v(\mathbb{D}, \mathbf{DblCat})_{d\ell} \to \mathbf{DblCat}/\mathbb{D}$$

  which is locally an isomorphism of double categories

$$\mathbb{H}\mathrm{om}_{d\ell}(F, G) \cong (\mathbf{DblCat}/\mathbb{D})\left(\int_{\mathbb{D}} F \to \mathbb{D}, \int_{\mathbb{D}} G \to \mathbb{D}\right).$$

Application I: Tricolimits in **2-Cat**

- For a 2-category $\mathcal{A}$ and a 2-functor $F\colon \mathcal{A} \to \mathbf{2\text{-}Cat}$, we construct a double index functor as follows. First take

$$\mathcal{A} \xrightarrow{F} \mathbf{2\text{-}Cat} \xrightarrow{\mathbb{V}} \mathbf{DblCat}_v$$

and then apply $\mathbb{V}$ to obtain:

$$\mathbb{V}(\mathcal{A}) \xrightarrow{\mathbb{V}(\mathbb{V}\circ F)} \mathbb{V}(\mathbf{DblCat}_v) \xrightarrow{\text{incl}} \mathbb{Q}(\mathbf{DblCat}_v).$$

- Applying the double Grothendieck construction gives us

$$\int_{\mathbb{V}\mathcal{A}} \mathbb{V}(\mathbb{V}\circ F) = \mathbb{V}\int_{\mathcal{A}} F$$

(as defined by Bakovic and Buckley)
- The functor $\mathbb{V}\colon \mathbf{2\text{-}Cat} \to \mathbf{DblCat}_v$ induces an isomorphism of 3-categories between **2-Cat** and its image in $\mathbf{DblCat}_v$.
- It follows that $\int_{\mathcal{A}} F$ is the **lax tricolimit** of F in **2-Cat**.

Application II: Categories of Elements

- For a functor $F\colon \mathbf{A} \to \mathbf{Set}$,

$$\operatorname{colim} F = \pi_0 \mathrm{El}\,(dF),$$

  where

$$\mathbf{A} \xrightarrow{F} \mathbf{Set} \xrightarrow{d} \mathbf{Cat}$$

  and $\mathrm{El}\,(dF)$ has objects (A, x) with $x \in F(A)$ and arrows $f\colon (A, x) \to (A', x')$ where $f\colon A \to A'$ with $F(f)(x) = x'$.
- This follows from the universal property of the elements construction as lax colimit by applying it to cones with discrete categories as vertex and using the adjunction $\pi_0 \dashv d$.

- We can apply the same paradigm to a functor $F\colon \mathcal{A} \to \mathbf{Cat}$ and use

$$\mathbf{Cat} \underset{\mathbb{V}}{\overset{\pi_0}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{DblCat}_v$$

  where the π_0 is taken with respect to horizontal arrows and cells to obtain a quotient of the vertical category of a double category.
- It follows from our Main Theorem that $\pi_0 \int_{\mathbb{H}\mathbf{A}} \mathbb{Q}(\mathbb{V} \circ F)$ gives the **strict 2-categorical colimit** of F.
- $\int_{\mathbb{H}\mathbf{A}} \mathbb{Q}(\mathbb{V} \circ F)$ is actually $\mathbb{E}l(F)$, introduced by Paré (1989): its double cells "(α, Φ)" are in this case given by 2-cells $\alpha\colon f \Longrightarrow f'$ in $\mathcal{A}$:

$$\begin{array}{ccc} (C,x) & \xrightarrow{(f,id)} & (D,y) \\ {\scriptstyle (id,\rho)}\big\downarrow\bullet & (\alpha,id) & \bullet\big\downarrow{\scriptstyle (id,\lambda)} \\ (C,x') & \xrightarrow[(f',id)]{} & (D,y') \end{array} \qquad\qquad \begin{array}{ccc} Ffx & \xrightarrow{id} & Ffx \\ {\scriptstyle (F\alpha)_x}\big\downarrow & id & \big\downarrow{\scriptstyle \lambda} \\ Ff'x & \xrightarrow[Ff'\rho]{} & Ff'x' \end{array}.$$

Application III: The double categorical wreath product

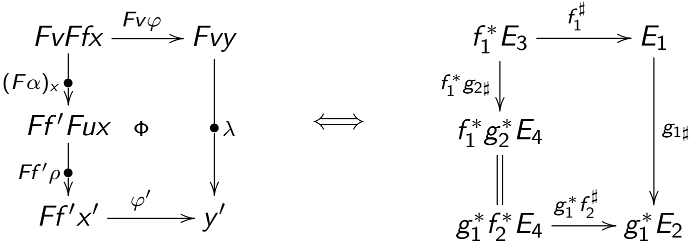
For a functor $F\colon \mathbf{A}^{\mathrm{op}} \to \mathbf{Cat}$, we consider:

$$\mathbf{A}^{\mathrm{op}} \xrightarrow{F} \mathbf{Cat} \xrightarrow{\mathbb{Q}} \mathbf{DblCat}_v \xrightarrow{(\ )^\wedge} \mathbf{DblCat}_v$$

where $\mathbb{E} \to \mathbb{E}^\wedge$ is the horizontal flip functor, and apply $\mathbb{Q}$ to all of this:

$$\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}((\mathbb{Q} \circ F)^\wedge) = F \wr F^{op}$$

as introduced by Myers (2020). In this case our Φ in (α, Φ) matches the basic diagram in his definition



Application IV (in progress): A tom Dieck Fundamental **Double** Groupoid

- Classically, the tom Dieck fundamental group is obtained as a quotient of a 2-categorical Grothendieck construction $\int_{\mathcal{O}_G} \Pi_X$.
- $\mathcal{O}_G$ is the 2-category of orbit types of G:
  - Objects: G/H where H is a closed subgroup of G;
  - Arrows: G-equivariant maps $a\colon G/K_1 \to G/K_2$ (generated by projections and conjugations); can also be viewed as points in $(G/K_2)^{K_1}$; they can also be viewed as elements of G: conjugation by a after a canonical projection.
  - 2-Cells: homotopy classes of paths in $(G/K_2)^{K_1}$.
- $\Pi_X(G/H) = \pi(X^H)$.

Work in Progress

- Extend the equivariant fundamental groupoid to an equivariant fundamental double groupoid:
  - Extend the orbit category to an orbit double category where the vertical arrows are given by certain paths in the topological group G.
  - The fundamental double groupoids on the fixed point spaces give rise to a vertical double functor and its doubly lax colimit is the equivariant fundamental double groupoid.
  - This provides a finer homotopy invariant than the tom Dieck fundamental groupoid.
- Extend the construction and the correspondence to double pseudo indexing functors $\mathbb{D} \to \mathbb{Q}(\mathbf{DblCat}_v)$.
- Describe the notion of fibration between double categories that characterizes the double functors of the form $\int_{\mathbb{D}} F \to \mathbb{D}$ and extend our results to a correspondence between suitable fibrations over $\mathbb{D}$ and (double pseudo) indexing functors $\mathbb{D} \to\!\!\!\!\!\shortmid\!\!\!\to \mathbf{DblCat}$.

Thank you!