MAKING CONCURRENCY FUNCTIONAL

Topos Institute Colloquium, 3 March 2022

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Two principal paradigms of computation

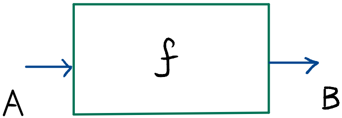
Functional: historically important paradigm, computation as functions; with interaction by function composition.
Many refinements: lenses, optics, combs, containers, dependent lenses, dependent optics, open games and learners, ...

Interactive: a more recent paradigm, computation as interacting processes. Many approaches, less settled, often syntax-driven.
Here a maths-driven foundation based on distributed/concurrent games based on event structures, with interaction by composition of strategies. Idea: types/constraints are games and programs/processes are strategies. $\rightsquigarrow$ *Syntax.*

This talk:

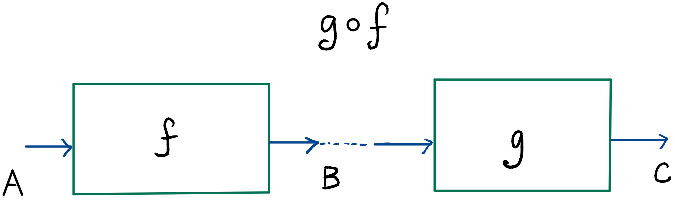
A bridge: how specialising games yields the functional paradigms.

Interaction via functions

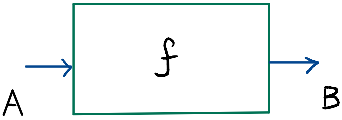


a function $f : A \to B$

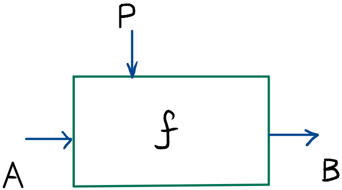
Interaction via functions



Interaction via functions



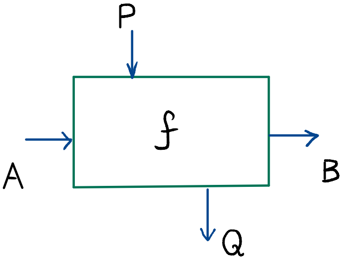
Interaction via functions



a parameterised function

$$f : A \otimes P \longrightarrow B$$

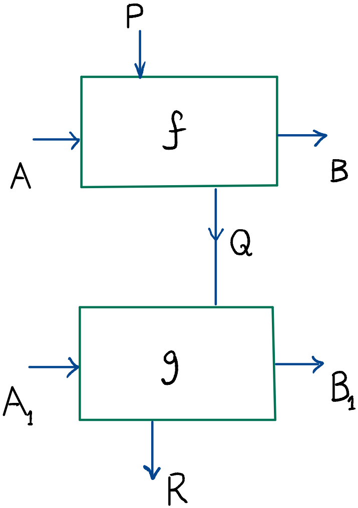
Interaction via functions



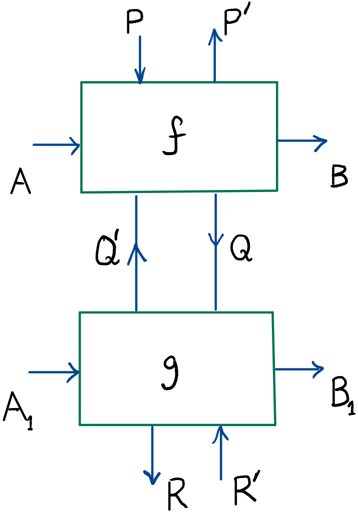
a parameterised function

$$f : A \otimes P \longrightarrow Q \otimes B$$

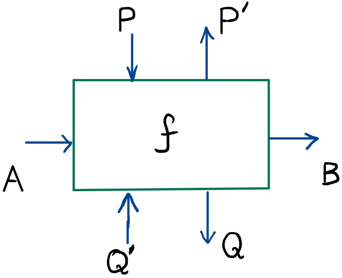
Interaction via functions



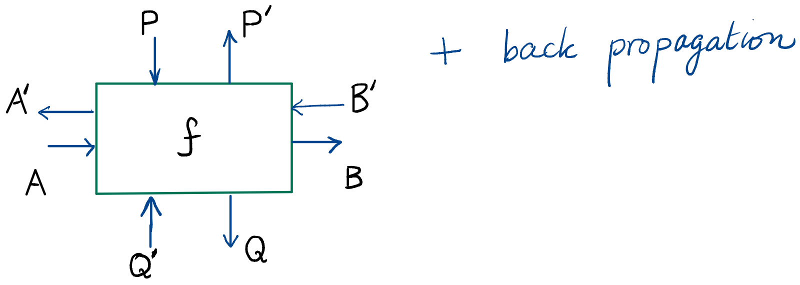
Interaction via functions



Interaction via functions



Interaction via functions



- Nature of functions? Generally need enrichments to functions which are partial, continuous, nondeterministic, probabilistic, quantum, smooth, ...
- Functions and their usual IO types can only give a static, partial picture of the dynamics of interaction. Dependent types can sometimes help, but ...
- Need a way to describe and orchestrate temporal pattern of interaction, its fine-grained dependencies and dynamic linkage.

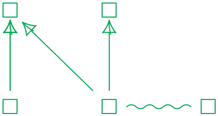
Event structures - of the simplest kind

Definition

An event structure comprises $(E, \leqslant, \#)$, consisting of a set of events E
- partially ordered by $\leqslant$, the causal dependency relation, and
- a binary irreflexive symmetric relation, the conflict relation,

which satisfy $\{e' \mid e' \leqslant e\}$ is finite and $e_1' \geqslant e_1 \# e_2 \leqslant e_2' \implies e_1' \# e_2'$.

Two events are concurrent when neither in conflict nor causally related.



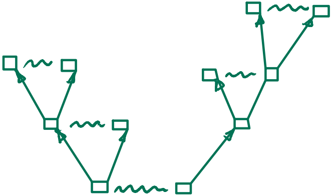
Definition

The configurations, $\mathcal{C}(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

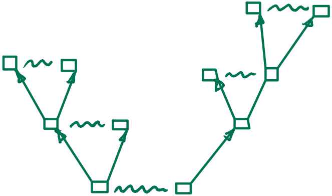
Consistent: don't have $e \# e'$ for any events $e, e' \in x$, and

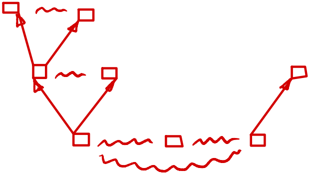
Down-closed: $e' \leqslant e \in x \implies e' \in x$.

Trees as event structures



Trees as event structures





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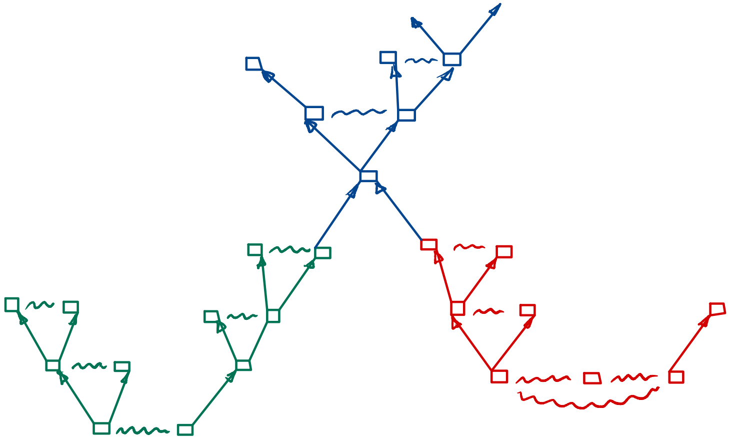


Games for interaction: the paradigm of Conway, Joyal

In 2-party games read Player vs. Opponent as Process vs. Environment.

Assume operations on (2-party) games:

Dual game $G^\perp$ *- interchange the role of Player and Opponent;*
Counter-strategy = strategy for Opponent = strategy for Player in dual game.

Parallel composition of games $G \| H$.

A strategy (for Player) from a game G *to a game* H *= strategy in* $G^\perp \| H$.
A strategy (for Player) from a game H *to a game* K *= strategy in* $H^\perp \| K$.

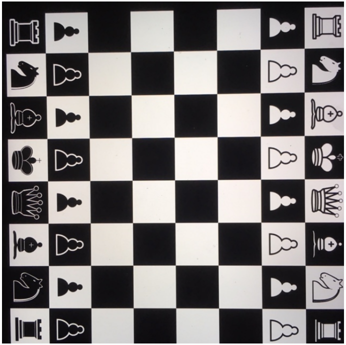
Compose by letting them play against each other in the common game H.

$\rightsquigarrow$ *has identity the Copycat strategy in* $G^\perp \| G$*, so from* G *to* G *...*

Copycat strategy illustrated

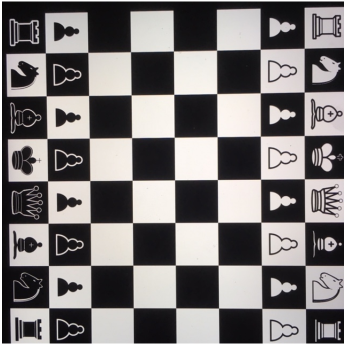
Chess, the game in which Player plays Black.

$Chess^{\perp}$



GM1 Player

Chess



Player GM2

Distributed games = Conway-Joyal on event structures

Games are represented by event structures with polarity, an event structure $(E, \leqslant, \#)$ *where events* E *carry a polarity, plus* $+$ *or* $-$ *for Player/Opponent. Assume race-free: no* immediate *conflict between Player and Opponent events.*

Dual, $B^\perp$*, of an event structure with polarity* B *is a copy of the event structure* B *with a reversal of polarities; this switches the roles of Player and Opponent.*

(Simple) Parallel composition: $A \| B$ *, by non-conflicting juxtaposition.*

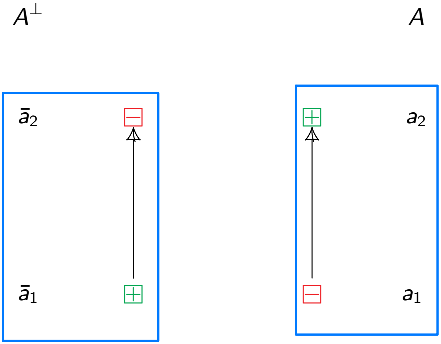
A strategy from a game A *to a game* B *is a strategy in* $A^\perp \| B$*, written*

$$\sigma : A \rightarrow\!\!\!\!+\!\!\!\!\!\!> B$$

But what's a strategy in game?

Roughly, a strategy (for Player) should be a choice of moves for Player together with their causal dependencies on Opponent moves.

Example: Copycat strategy from A to A



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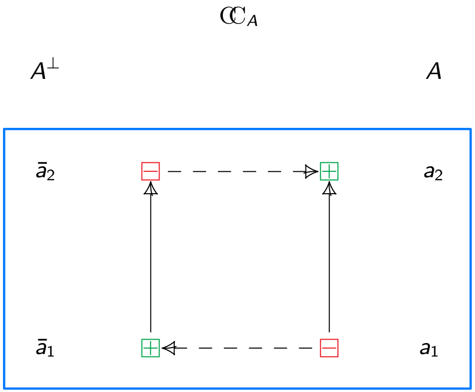
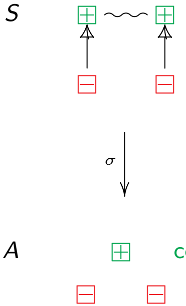


In general a strategy in a game A comprises

an event structure S and a function on events $\sigma : S \to A$,
so for all configurations x of S, its image σx is a configuration of A and
if $s_1, s_2 \in x$ and $\sigma(s_1) = \sigma(s_2)$ then $s_1 = s_2$.

e.g.

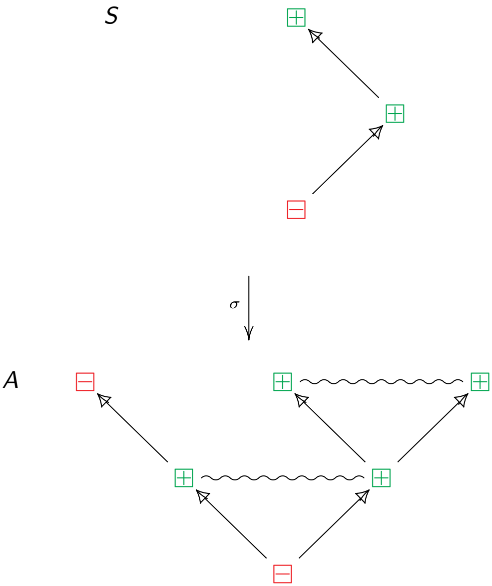


configurations of S = "states of play"

configurations of A = "positions of the game,"

which is (1) receptive and (2) innocent:
(1) any Opponent move at a position in A is allowed at the state of play in S;
(2) in S the only additional causal dependencies are $\boxminus \rightarrowtriangle \boxplus$.

When games are trees:



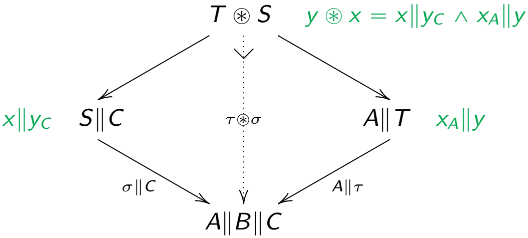
The strategy: Player takes the initiative.

Composition of strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$

To compose

synchronise complementary moves over common game B via pullback.

A configuration $y \circledast x$ of $T \circledast S$ glues x and y together over common part $x_B = y_B$ provided no causal loops ensue:



$x_A \| x_B \| y_C = x_A \| y_B \| y_C$

Composition of strategies $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$

To compose

$$\begin{array}{ccc} S & & T \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ A^{\perp} \| B & & B^{\perp} \| C \end{array}$$

synchronise complementary moves over common game B (via pullback); then hide synchronisations (via partial-total factorisation):

$$\begin{array}{ccccc} \textit{before hiding} & T \circledast S & \cdots\cdots\rightharpoonup & T \odot S & \textit{after hiding} \\ & \downarrow{\scriptstyle \tau \circledast \sigma} & & \downarrow{\scriptstyle \tau \odot \sigma} & \\ & A^{\perp} \| B \| C & \cdots\cdots\rightharpoonup & A^{\perp} \| C & \end{array}$$

Theorem (Rideau, W)

Conditions of receptivity and innocence on a strategy are precisely those needed to make copycat identity w.r.t. composition.

For copycat to be identity w.r.t. composition

a strategy in a game A has to be $\sigma : S \to A$, a total map of event structures with polarity, such that

(i) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a unique $x' \in \mathcal{C}(S)$ s.t. $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ \sigma \downarrow & & \vdots\ \sigma \\ \sigma x & \subseteq^- & y\,, \end{array}$$

A strategy should be receptive to Opponent moves allowed by the game.

(ii) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ s.t. $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x' & \cdots\subseteq\cdots & x \\ \sigma\ \vdots & & \downarrow \sigma \\ y & \subseteq^+ & \sigma x\,. \end{array}$$

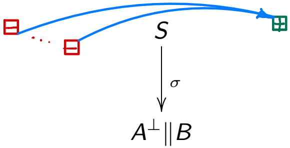
A strategy should only adjoin immediate causal dependencies $\boxminus \rightarrow \boxplus$.

$\rightsquigarrow$ compact-closed bicategory of distributed games and strategies.

Special case: Gérard Berry's dI-domains and stable functions

A concurrent strategy is deterministic when conflicting behaviour of Player implies conflicting behaviour of Opponent.

Let A and B be purely Player games.
A strategy from A to B is a strategy in $A^{\perp} \| B$:



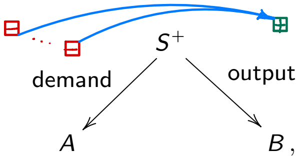
Deterministic strategies $\sigma : A \rightarrow\!\!\!+\!\!\!\!> B$ correspond to stable functions $f : (\mathcal{C}(A), \subseteq) \rightarrow (\mathcal{C}(B), \subseteq)$ between dI-domains. They have a function space $[A \rightarrow B]$ w.r.t. product $\|$. A stable function preserves least upper bounds of directed sets and meets of compatible elements.

Theorem

There is an equivalence between deterministic strategies between purely Player games and Gérard Berry's dI-domains and stable functions.
The equivalence restricts to one is with Jean-Yves Girard's coherence spaces when causal dependencies are the trivial identity relation.

Special case: stable spans

Let A and B be purely Player games. Strategies, possibly nondeterministic, $\sigma : A \rightarrow\!\!\!\!+\!\!\!\!\!\!\rightarrow B$ correspond to stable spans



roughly, nondeterministic stable functions; they are (special) profunctors. Stable spans have been central in providing a compositional model for nondeterministic dataflow; the feedback of nondeterministic dataflow is given by the trace of strategies. Stable spans have a function space $[A \multimap B]$ w.r.t. $\|$.

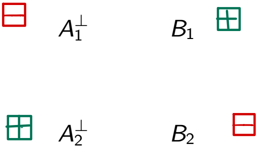
Theorem

There is an equivalence between strategies between purely Player games and stable spans between event structures.

Girard's Geometry of Interaction (the nature of proofs as networks)

A GoI game A comprises a parallel composition $A_1 \| A_2$ where A_1 is a purely Player game and A_2 is a purely Opponent game.

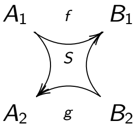
A strategy σ from a GoI game $A := A_1 \| A_2$ to a GoI game $B := B_1 \| B_2$ is a strategy in $A^\perp \| B$, *i.e.*



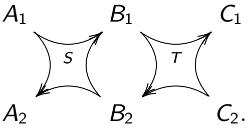
so a strategy $A_1 \| B_2^\perp \longrightarrow\!\!\!\!\!+ A_2^\perp \| B_1$ between purely Player games.
A deterministic strategy from A to B corresponds to a pair of stable functions

$$f : \mathcal{C}(A_1) \times \mathcal{C}(B_2) \to \mathcal{C}(A_2) \text{ and } g : \mathcal{C}(A_1) \times \mathcal{C}(B_2) \to \mathcal{C}(B_1)\,,$$

summarised by



with composition



We recover Abramsky and Jagadeesan's GoI construction, but now starting from *stable* domain theory. Applications: optimal reduction, token machines

Extensions - winning conditions and imperfect information

A *winning condition* on a game A specifies those of its configurations which are a win for Player.
A strategy in A is *winning* (for Player) if in any maximal play for Player results in a winning configuration of A.
A strategy from A to B, *i.e.* in $A^{\perp} \| B$, is *winning* if, in any maximal play for Player, a win in A implies a win in B.
Winning strategies compose.

Imperfect information via an access order $(\Lambda, \preceq)$ on moves of games.
Idea: moves have an access level and can only depend on moves $\preceq$-lower.
Causal dependency of the game and strategy must respect $\preceq$:

$$\begin{array}{ccc} S & & \\ \downarrow \sigma & & \\ A & \xrightarrow{\lambda} & (\Lambda, \preceq) \end{array} \qquad \begin{array}{c} s' \leqslant_S s \\ \text{implies} \\ \lambda\sigma(s') \preceq \lambda\sigma(s) \end{array}$$

Such Λ-strategies compose.

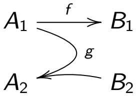
Gödel's dialectica interpretation

A *dialectica game* *is a GoI game* $A = A_1 \| A_2$ *with winning conditions and access levels*

$$1 \prec 2$$

with Player moves A_1 *assigned* 1 *and Opponent moves* A_2 *assigned* 2*.*

A deterministic strategy between dialectica games, from A to B, is a lens:



It's winning means

$$W_A(x, g(x, y)) \implies W_B(f(x), y)\,,$$

for all configurations x of A_1 and y of B_2.

Deterministic strategies between dialectica games coincide with Gödel's dialectica interpretation of proofs in arithmetic as higher-order functions [Gödel, de Paiva, Hyland]. Applications: proof mining [Kreisel, Kohlenbach].

Girard's variant and Combs

Girard's variant [de Paiva]: Just changing Λ to the discrete order

$$1 \bullet \quad \bullet 2$$

enforces *non-signalling*, rather than the *one-way signalling* of dialectica games, between moves of the two different access levels.

A deterministic strategy $A \longrightarrow\!\!\!\!\!\!\!+\;\;\; B$ now corresponds to a pair of stable functions

$$A_1 \xrightarrow{f} B_1$$
$$A_2 \xleftarrow[g]{} B_2.$$

Combs of quantum architecture arise as strategies between "comb games," comprising n-fold alternating-polarity, parallel compositions

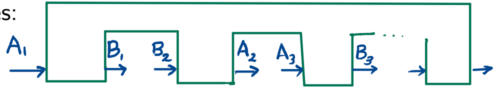
$$A_1 \| A_2 \| \cdots \| A_n$$

of purely Player and purely Opponent games over access levels

$$1 \prec 2 \prec \cdots \prec n$$

A strategy between comb games:

$$\sigma : A \longrightarrow\!\!\!\!\!\!\!+\;\;\; B$$

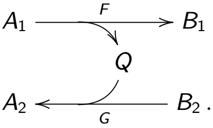


Optics

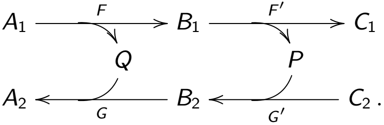
General, possibly nondeterministic, strategies between dialectica games $\sigma : A \mathrel{\rightarrow\!\!\!\!\!+\!\!\!>} B$ correspond to optics built from stable spans

$$F : A_1 \mathrel{\rightarrow\!\!\!\!\!+\!\!\!>} B_1 \| Q \quad \text{and} \quad G : Q \| B_2^{\perp} \mathrel{\rightarrow\!\!\!\!\!+\!\!\!>} A_2^{\perp}$$

as the composite



Composition of strategies coincides with composition of optics



Strategies $\sigma : A \mathrel{\rightarrow\!\!\!\!\!+\!\!\!>} B$ between dialectica games, so optics on stable spans, correspond to stable spans of type $A_1 \multimap B_1 \| [B_2 \multimap A_2]$.

Containers (data structures where "shapes" index "positions")

A container game is a game of imperfect information A w.r.t. access levels $1 \prec 2$; each Player move of A assigned 1 and each Opponent move 2.

The only causal dependencies in A relating moves of different polarities: $\boxplus < \boxminus$
The game A comprises an initial Player part A_1 followed by a dependent Opponent part A_2; its configurations have form $x \cup y$ where x comprises solely Player moves and y solely Opponent moves, dependent on x.
Hence the container game A corresponds to a dependent type $\Sigma_{x:A_1} A_2(x)$.

A deterministic strategy between container games $\sigma : A \twoheadrightarrow B$ corresponds to a dependent lens, a pair of stable functions

$$f : [A_1 \to B_1] \text{ and } g : \Pi_{x:A_1} [B_2(f(x)) \to A_2(x)] .$$

I.e. to an element of type

$$\Sigma_{f:[A_1 \to B_1]} \Pi_{x:A_1} [B_2(f(x)) \to A_2(x)] ;$$

so, less standardly, to an element of the isomorphic type

$$\Pi_{x:A_1} \Sigma_{y:B_1} [B_2(y) \to A_2(x)] .$$

Dependent optics (new?)

General, possibly nondeterministic, strategies between container games $\sigma : A \longrightarrow\!\!\!\!\!+\!\!\!> B$ correspond to "dependent optics" of type

$$\mathrm{dOp}[A, B] = \Pi^s_{x:A_1} \Sigma_{y:B_1} \; [B_2(y) \multimap A_2(x)] \,,$$

where Π^s is a dependent product of stable spans.
Dependent optics compose by

$$\circ : \mathrm{dOp}[B, C] \parallel \mathrm{dOp}[A, B] \longrightarrow\!\!\!\!\!+\!\!\!> \mathrm{dOp}[A, C]$$

described, a little informally, as

$$G \circ F =_{\mathrm{def}} \lambda x : A_1 . \;\; \mathrm{let}\ (y, F') \Leftarrow F(x)\ \mathrm{in}$$
$$\mathrm{let}\ (z, G') \Leftarrow G(y)\ \mathrm{in}\ (z, F' \odot G')$$

where $F' \odot G'$ is the composition of stable spans

$$G' : [C_2(z) \multimap B_2(y)] \;\; \text{and} \;\; F' : [B_2(y) \multimap A_2(x)] \,.$$

Functional paradigms that arise as special strategies inherit the enrichments of strategies, probabilistic, quantum, real-number, ...

Enrichments via parameterised maps

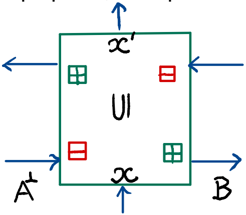
Games and strategies support enrichments to: probabilistic strategies, also with continuous distributions; quantum strategies; smooth functions to support differentiation.

Recent realisation: all enrichments can be achieved uniformly by the same construction, using parameterised "functions" [Clairambault, de Visme, W].

W.r.t. a symmetric monoidal category $(\mathcal{M}, \otimes, I)$, *e.g.* $([0,1], \cdot, 1)$ or CPM,

1. moves of a game are assigned objects in $\mathcal{M}$;

2. intervals $x \subseteq x'$ of finite configurations of S in a strategy are assigned parameterised maps over $\mathcal{M}$, the polarity of events deciding which way the parameter maps point, as input or output:



The events, their dependencies and polarities, orchestrate the functional dependency and dynamic linkage in composing *enriched* strategies.

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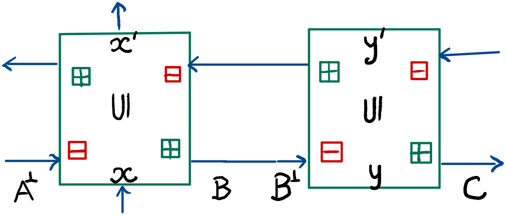


The events, their dependencies and polarities, orchestrate the functional dependency and dynamic linkage in composing *enriched* strategies.

Conclusion

Functional approach can help tame wild world of concurrent interaction through providing simpler models.
But adapting functions to interaction often requires considerable ingenuity, especially when requiring enrichments, *e.g.* probabilistic, quantum, real-number.

Distributed games and strategies provide a broad general context for interaction which can be specialised to functional paradigms;
also in providing enrichments to probabilistic, quantum and real number computation *etc.*

For more detail: `http://arxiv.org/abs/2202.13910`

THANK YOU!

Enrichments via parameterised category

Games and strategies support enrichments: to probabilistic, also with continuous distributions, quantum strategies, smooth functions to support differentiation. All enrichments are achieved uniformly by the following construction.

Definition

Let $(\mathcal{M}, \otimes, \mathcal{I})$ be a symmetric monoidal category.
Define the parameterised category $\mathrm{Para}^{\mathrm{IO}}(\mathcal{M})$ to have objects those of $\mathcal{D}$ and maps $(P, f, Q) : X \to Y$ consisting of $f : X \otimes P \to Q \otimes Y$ in $\mathcal{M}$.
Let $(P, f, Q) : X \to Y$ and $(R, g, S) : Y \to Z$ be maps in $\mathrm{Para}^{\mathrm{IO}}(\mathcal{M})$. Their composition $(R, g, S) \circ (P, f, Q) : X \to Z$ is given by
$(R, g, S) \circ (P, f, Q) = (P \otimes R,\ (Q \otimes g) \circ (f \otimes R),\ Q \otimes S)$.

Examples of $\mathcal{M}$

The monoid of Reals, or unit interval, under multiplication.
Measurable spaces with Markov kernels.
Completely Positive Maps.
Euclidean spaces with smooth maps.

Enrichments of Games and Strategies w.r.t. a symmetric monoidal category

Definition

A $\mathcal{M}$-game is a game A together with a function H from the events A to objects of $\mathcal{M}$. Extend H to finite subsets of events. Given a finite subset $X \subseteq_{\mathrm{fin}} A$ define $H(X) =_{\mathrm{def}} \bigotimes_{a \in X} H(a)$.

Definition

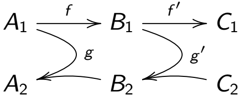
A $\mathcal{M}$-strategy in a $\mathcal{M}$-game (A, H) is a strategy $\sigma : S \to A$ together with a functor

$$Q : (\mathcal{C}(S), \subseteq) \to \mathrm{Para}^{\mathrm{IO}}(\mathcal{M})$$

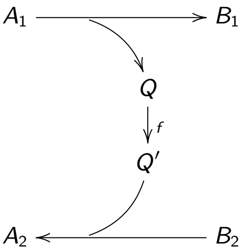
taking $x \subseteq y$ to a map $Q(x \subseteq y) = (H(\sigma(y \backslash x)^-), f, H(\sigma(y \backslash x)^+))$; we insist the map f is an isomorphism in the case where $x \subseteq^- y$.

The monoid of Reals, or unit interval, under multiplication
$\rightsquigarrow$ probabilistic strategies.
Measurable spaces with Markov kernels
$\rightsquigarrow$ probabilistic strategies with continuous distributions.
Completely Positive Maps
$\rightsquigarrow$ (functorial) quantum strategies (refining those of POPL'19).
Euclidean spaces with smooth maps
$\rightsquigarrow$ strategies assigning smooth maps to Player moves.

Composing lenses



The optic equivalence relation



Enrichments via parameterised maps

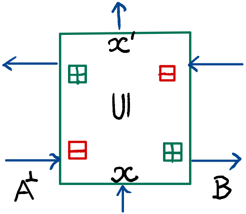
Games and strategies support enrichments to: probabilistic strategies, also with continuous distributions; quantum strategies; smooth functions to support differentiation.

Recent realisation: all enrichments can be achieved uniformly by the same construction, using parameterised "functions" [Clairambault, de Visme, W].

W.r.t. a symmetric monoidal category $(\mathcal{M}, \otimes, I)$, *e.g.* $([0,1], \cdot, 1)$ or CPM,

1. moves of a game are assigned objects in $\mathcal{M}$;

2. intervals $x \subseteq x'$ of finite configurations of S in a strategy are assigned parameterised maps over $\mathcal{M}$, the polarity of events deciding which way the parameter maps point, as input or output:



The events, their dependencies and polarities, orchestrate the functional dependency and dynamic linkage in composing *enriched* games and strategies.

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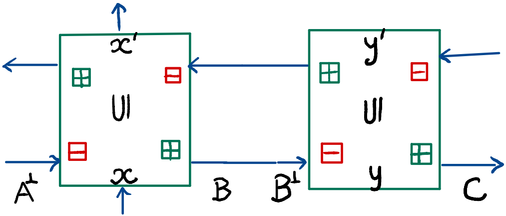

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