Lessons from failing distributive laws

Maaike Zwart

IT University of Copenhagen
maaike.annebeth@gmail.com / www.maaikezwart.com

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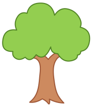



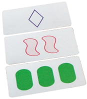

Overview

- Introduction
  - Motivation: monads and monad compositions
- How to: No-go Theorems
  - including proof of 50 year old problem!
- A crucial step
- What I am doing now
- Conclusion

Motivation: monads and monad compositions

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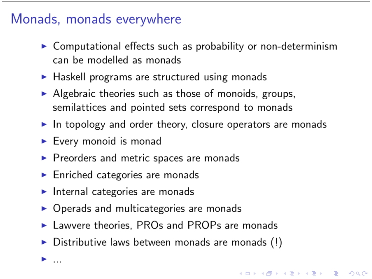

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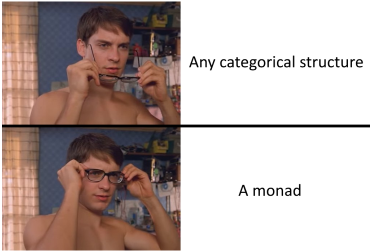

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Monads, monads everywhere

- Computational effects such as probability or non-determinism can be modelled as monads
- Haskell programs are structured using monads
- Algebraic theories such as those of monoids, groups, semilattices and pointed sets correspond to monads
- In topology and order theory, closure operators are monads
- Every monoid is monad
- Preorders and metric spaces are monads
- Enriched categories are monads
- Internal categories are monads
- Operads and multicategories are monads
- Lawvere theories, PROs and PROPs are monads
- Distributive laws between monads are monads (!)
- ...

Compositions of monads allow simultaneous modelling of multiple computational aspects.

Monads: Monoids in the category of endofunctors

A monad is a triple $\langle T, \eta, \mu \rangle$, with T an endofunctor and $\eta : 1 \Rightarrow T$, $\mu : TT \Rightarrow T$ natural transformations, such that:

$$
\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, "Id"] & TT \arrow[d, "\mu"] \\
TT \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
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Examples:

- List, $[x]$, ++
- Multiset/Bag
- Powerset
- Distribution
- Exception
- Writer
- Reader
- State

Composing Monads

- Find η^{TS}, μ^{TS} such that $\langle TS, \eta^{TS}, \mu^{TS} \rangle$ is a monad.

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- If λ is a *distributive law*, then the above choices form a monad.

Composing Monads with Distributive Laws

The following composite is a monad *- Beck 1969*.

$$\langle TS, \eta^T\eta^S, \mu^T\mu^S \cdot T\lambda S\rangle,$$

where $\lambda : ST \to TS$ is a natural transformation satisfying the following axioms.

$$\begin{tikzcd} & T \arrow[dl, "\eta^S T"'] \arrow[dr, "T\eta^S"] & \\ ST \arrow[rr, "\lambda"] & & TS \end{tikzcd} \qquad \begin{tikzcd} SST \arrow[r, "S\lambda"] \arrow[d, "\mu^S T"'] & STS \arrow[r, "\lambda S"] & TSS \arrow[d, "T\mu^S"] \\ ST \arrow[rr, "\lambda"] & & TS \end{tikzcd}$$

$$\begin{tikzcd} & S \arrow[dl, "S\eta^T"'] \arrow[dr, "\eta^T S"] & \\ ST \arrow[rr, "\lambda"] & & TS \end{tikzcd} \qquad \begin{tikzcd} STT \arrow[r, "\lambda T"] \arrow[d, "S\mu^T"'] & TST \arrow[r, "T\lambda"] & TTS \arrow[d, "\mu^T S"] \\ ST \arrow[rr, "\lambda"] & & TS \end{tikzcd}$$

Examples

There is a distributive law for List over Powerset. It works like the famous ‘times over plus’ distributivity:

$$(a + b) * c = a * c + b * c$$
$$[\{a, b\}, \{c\}] \mapsto \{[a, c], [b, c]\}$$

Many more work like this:

- Multiset over itself
- List over Multiset
- Multiset over Powerset
- ...

That sounds easy, but...

Problem:

- Distributive laws are hard to find.
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- No-go theorems for distributive laws.

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My weapon of choice:

- Algebra.

A quick reminder: algebraic theories

Algebraic theory:

- Signature Σ and a set of variables give *terms*.
- Axioms E and equational logic give equivalence of terms.

Monoids:

$$\Sigma = \{1^{(0)}, *^{(2)}\}$$
$$\begin{aligned} E = \{&1 * x = x = x * 1, \\ &(x * y) * z = x * (y * z)\} \end{aligned}$$

Abelian groups:

$$\Sigma = \{0^{(0)}, -^{(1)}, +^{(2)}\}$$
$$\begin{aligned} E = \{&0 + x = x = x + 0, \\ &(x + y) + z = x + (y + z), \\ &x + y = y + x, \\ &x + (-x) = 0 = (-x) + x\} \end{aligned}$$

The algebraic equivalent of distributive laws

Monads	$\iff$	Algebraic theories
Distributive laws	$\iff$	

The algebraic equivalent of distributive laws

Monads	$\Longleftrightarrow$	Algebraic theories
Distributive laws	$\Longleftrightarrow$	Composite theories *- Piróg and Staton 2017*.

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Example: Composing Abelian groups and Monoids: Rings!

$$\begin{aligned}\Sigma^R &= \Sigma^A \uplus \Sigma^M \\ &= \{0^{(0)}, 1^{(0)}, -^{(1)}, +^{(2)}, *^{(2)}\} \\ E^R &= E^A \cup E^M \cup \\ &\quad \{a * (b + c) = (a * b) + (a * c) \\ &\quad (a + b) * c = (a * c) + (b * c)\}\end{aligned}$$

The algebraic equivalent of distributive laws

Monads $\iff$ Algebraic theories
Distributive laws $\iff$ Composite theories *- Piróg and Staton 2017*.

- Terms can be separated

$$a * (b + c) = (a * b) + (a * c)$$

- Equality preservation of component theories (*essential uniqueness*) - only for two separated terms!

$$(a * b) + c =_R c + (a * b)$$
$$\Leftrightarrow \; x + c =_A c + x \text{ and } a * b =_M a * b$$

My strategy: no-go theorems for distributive laws

Using composite theories:

- Choose theories to compose: $\mathbb{T} \circ \mathbb{S}$.

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- List equations in the proof.
- $\Rightarrow$ No-go theorem.

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Proof that in Rings (Abelian groups after Monoids), $x * 0 = 0$

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equality holds in component theories.

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Gnirs as a first counterexample

There is no composite theory of Monoids after Abelian groups.

Proof:

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$$\begin{aligned}
& x \\
& \{unit\} \\
= \;& x + 0 \\
& \{inverse\} \\
= \;& x + (1 + (-1)) \\
& \{associativity\} \\
= \;& (x + 1) + (-1) \\
& \{x + 1 = 1\} \\
= \;& 1 + (-1) \\
& \{inverse\} \\
= \;& 0
\end{aligned}$$

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Jon Beck 108

The composite $\mathbf{ST}$ is the free ring triple. XST is the polynomial ring $\mathbf{Z}[X]$ with the elements of X as noncommuting indeterminates.

The canonical diagram of adjoint functors is:

[...]

The scheme is: the distributive law $\ell\colon TS \longrightarrow ST$ produces the adjoint square, which, being distributive (Section 3), induces a distributive law $\lambda\colon \mathbf{G}_{\mathbf{Ab}}\mathbf{G}_{\mathbf{Mon}} \longrightarrow \mathbf{G}_{\mathbf{Mon}}\mathbf{G}_{\mathbf{Ab}}$, where $G_{\mathbf{Mon}} = \widetilde{U^T}\widetilde{F^T}$, $G_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbf{T}}(F^{ST}, \ )\otimes_{\mathbf{T}} F^{ST}$. This λ is that employed by Barr in his *Composite cotriples*, this volume (Theorem 4.6).

A distributive law $ST \longrightarrow TS$ would have the air of a universal solution to the problem of factoring polynomials into linear factors. This suggests that the composite TS has little chance of being a triple.

(2) Constants. Any set C can be interpreted as a triple in the category of sets, $\mathbf{A}$, via the coproduct injection and folding map $X \longrightarrow C + X, C + C + X \longrightarrow C + X$. $\mathbf{A}^{C+(\ )}$ is the category of sets with C as constants. For example, if $C = 1$, $\mathbf{A}^{1+(\ )}$ is the category of pointed sets.

Some examples from various No-Go Theorems

Powerset $\circ$ Abelian groups

List $\circ$ Powerset

List^2

Multiset^3

Exception $\circ$ List

Multiset $\circ$ Rings

Powerset^2

Distribution^2

The crucial step

Composite theories give 2 properties:

- Separation
  Start with term x that is **not** separated:
  $x = ?$, where ? is separated.
- Essential Uniqueness
  Needs equality between two separated terms.

TODO: obtain a separated term from x.
Previously:

$$x * 0 \to 1 * 0 \to 0$$

Using, for all x:

$$1 * x = x$$

The crucial step

Trick: shrinking terms to variables or constants creates separated terms.

$$
\begin{tikzcd}
 & s(t(x,y),z) \arrow[dl] \arrow[d] \arrow[dr] & \\
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Units: $x + 0 = x$
Idempotence: $x * x = x$
absorption: $x \wedge (x \vee y) = x$
inverse: $x + (-x) = 0$

$\vdots$

The crucial step

Conjecture 1. *Any theorem that proves the non-existence of a distributive law will involve at least one monad that is presented by an algebraic theory $\mathbb{S}$ for which the following axiom holds:*

- $\mathbb{S}$ *has an n-ary term s ($n \geq 2$), for which there is a substitution $f : \mathrm{var}(s) \to \mathbb{S}$ such that for any $x \in \mathrm{var}(s)$:*

$$\Gamma \vdash s[f(y)/y \neq x] =_{\mathbb{S}} x.$$

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- ...
- $\Rightarrow$ General Theory

Conclusion & What is next

Conclusion and peek into the future:

- Not all monads compose via a distributive law.
- Algebra provides method to prove counterexamples, which can be generalised to no-go theorems.
- Reducing a term to a variable is a key property for no-go theorems.

Conclusion & What is next

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- Not all monads compose via a distributive law.
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What I am going to do:

- Combine algebraic effects with guarded recursion.
  - List and Multiset: done
  - Powerset might not be possible.
  - upcoming: reader, state, ...

References

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