Compact totally separated types in univalent mathematics

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The problem addressed here

$2 = \{0,1\}$
$\cong 1 + 1 \neq \Omega$

Given a set X and $p : X \to 2$,

- either exhibit $x \in X$ such that $p(x) = 0$ (a root of p)
- or else determine that p has no root.

$\boxed{\text{For which sets } X \text{ can this be done?}}$

- In terms of computation, this is an exhaustive search problem. Can we exhaustively search an infinite set mechanically?
- In terms of logic, this is a choice problem. can we prove non-trivial instances of choice?
- In terms of topology, this turns out to be a compactness problem.

Our type theory

(For some results we generalize to inductive-recursive types)

Martin-Löf Type Theory

MLTT $\quad \mathbb{0}, \mathbb{1}, \mathbb{N}, +, \times, \Sigma, \Pi, \mathrm{Id}, \mathcal{U}, W$

+

univalence (so in particular we have functional and propositional extensionality)

+

quotients ($\Longleftrightarrow$ propositional truncations + set replacement)

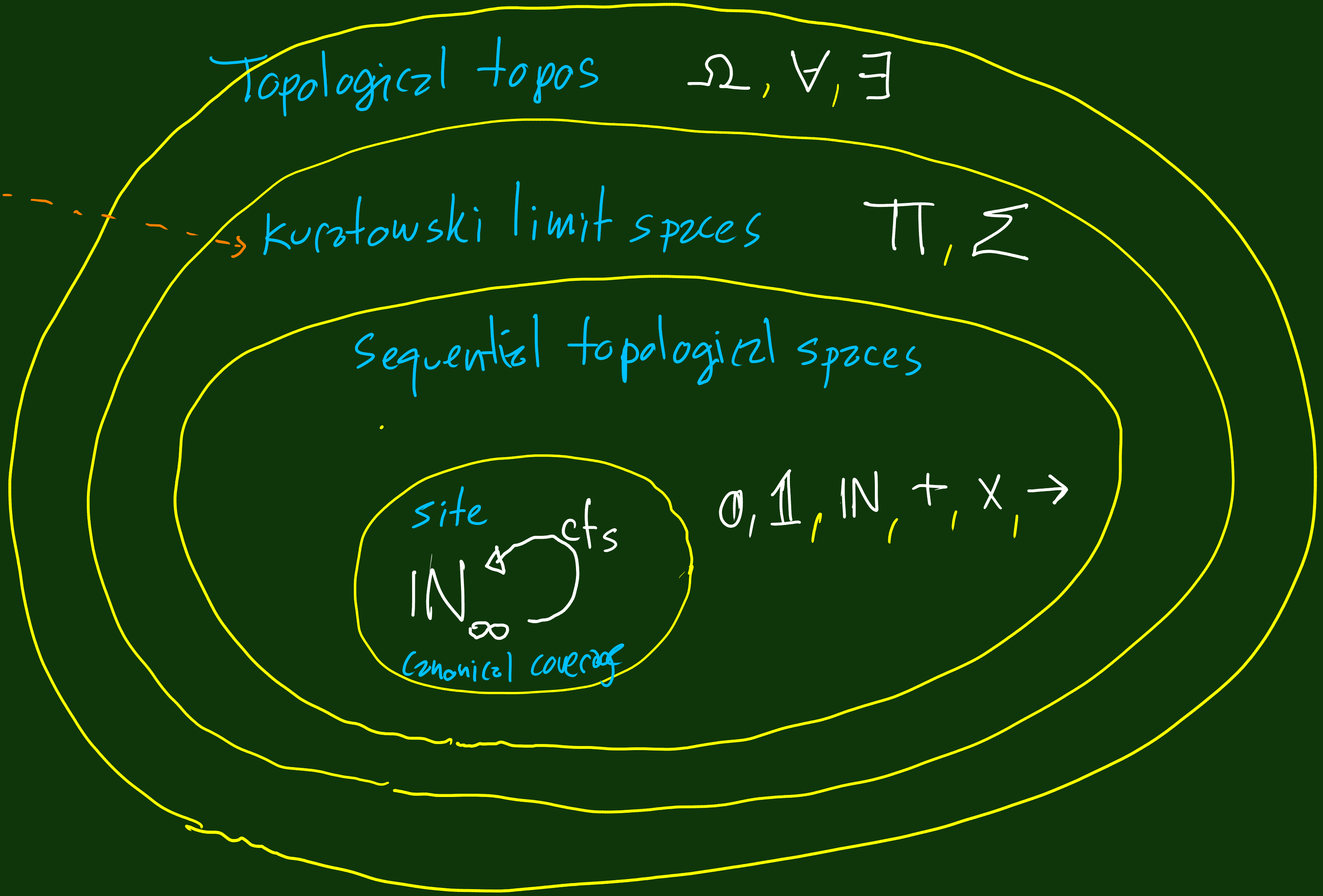
We reason constructively, so:
Our results hold in all models.

Many models

- Types are sets.
- Types are spaces. • Types are homotopy types.
- Types are "sets with computational structure" (realizability).
- Types are the objects of a topos.

One particular model plays a guiding role

concrete sheaves



Johnstone
1979

Examples of MLTT definable objects in that topos

- $\mathbb{N}$ and $\mathbb{2} \stackrel{\text{def}}{=} \mathbb{1} + \mathbb{1}$ get the discrete topology.
- $\mathbb{N} \to \mathbb{2}$ is the Cantor space, and $\mathbb{N} \to \mathbb{N}$ is the Baire space.
- $\mathbb{N}_\infty \stackrel{\text{def}}{=} \sum_{\alpha : \mathbb{N} \to \mathbb{2}} \prod_{i : \mathbb{N}} \alpha_i \geq \alpha_{i+1}$ is the one-point compactification of $\mathbb{N}$.

$\underline{n} \stackrel{\text{def}}{=} 1^n 0^\omega$

$\infty \stackrel{\text{def}}{=} 1^\omega$

$\mathbb{N} \hookrightarrow \mathbb{N}_\infty$

$n \longmapsto \underline{n}$

- $\sum_{x : \mathbb{N}_\infty} ((x = \infty) \to \mathbb{2})$ looks like this

This is compact T_1 but not Hausdorff.

We have $\{0, 1, \ldots, \infty_0\} \cap \{0, 1, \ldots, \infty_1\} = \mathbb{N}$

compact

not compact

Mathematical expression of the problem in our system

We can pick a root of p if it has any.

$$\prod p : X \to 2, \left(\sum x : X, \ p\,x = 0\right) + \underbrace{\left(\prod : X, \ p\,x = 1\right)}_{\neg \sum x : X,\ p\,x = 0}$$

- Stronger than excluded middle.
- We are making a choice.

We have Σ rather than $\exists$.

We ask which types X satisfy this choice principle.

Definition. We call such types compact.

$\boxed{\text{All types are compact} \iff \text{global choice holds}}$

Global choice: We can choose a point of every non-empty type.

$$\prod_{X : \mathcal{U}} \underbrace{\neg\neg X}_{X \text{ is non-empty}} \longrightarrow X$$

E.g. Voevodsky's model of simplicial sets

- stronger than choice, which is consistent with univalence.
- Contradicts univalence.
- But there are plenty of compact types in HoTT/UF.
- The ones we can construct are all equipped with well-orders.

Ordinals

X equipped with a proposition-valued relation $<$ satisfying

1. $<$ is transitive
2. If two points have the same precessors then they are equal.
3. $<$ satisfies transfinite induction

$$\left(\prod x : X \left(\prod y : X,\ y < x \to P y\right) \to P x\right) \to \prod x : X,\ P x$$

- X is automatically a set by (2) (its identity types are propositions)
- Trichotomy $X < Y$ or $X = Y$ or $X > Y$ is equivalent to excluded middle.
- But there are lots of well-ordered types that are trichotomous

The large type of all small ordinals

Univalence implies that this type

1. Is a (large) ordinal,
2. Has suprema of arbitrary small families.

Functions $p : X \to 2$

They classify complemented subtypes of X.

$$X \simeq \left(\sum_{x:X} p\,x = 0\right) + \left(\sum_{x:X} p\,x = 1\right).$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook, "\text{complemented}"'] & \mathbb{1} \arrow[d, "0"] \\ X \arrow[r, "p"'] & 2 \end{tikzcd}$$

In models:

- Topological topos. They classify clopen subspaces.
- Realizability toposes. They classify decidable subobjects.

complemented c.e. subobjects with c.e. complement.

Totally separated types

Recall

Definition. A type X is called *compact* if

$$\prod p : X \to 2,\ \left(\sum x : X,\ px = 0\right) + \left(\prod x : X,\ px = 1\right).$$

This definition is not good unless there are plenty of maps $X \to 2$.

Definition. A type X is called *totally separated* if

$$\prod x, y : X,\ \left(\prod p : X \to 2,\ px = py\right) \to x = y.$$

In the topological topos. The clopens separate the points.

(Topological notion with the same name.)

Some facts

1. Totally separated types are sets (their identity types are propositions)

2. They form an exponential ideal (more generally a "Π-ideal") and are closed under $+$, $\times$, retracts and include $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbb{N}, \mathbb{N}_\infty$ and all discrete types (those with decidable equality).

3. They are not closed under Σ in general.

Example. In the topological topos, the type $\Sigma_{x:\mathbb{N}_\infty}(x=\infty \to \mathbf{2})$ is not totally separated.

not totally separated

totally separated

(Compact totally separated spaces are Hausdorff. Also known as Stone spaces.)

4. Define the simple types to be the smallest collection of types including $\mathbb{0}$, $\mathbb{1}$, $\mathbb{N}$ and closed under $\times$, $+$, $\to$.

The simple types are all totally separated (by (2) above).

5. In the topological topos, a subtype of a simple type is compact in the above type-theoretic sense iff it is compact in the topological sense.

In this case the inclusion is a section and hence the subtype is itself totally separated.

Counter example

A so-called constructive taboo.

- The set $\mathbb{N}$ of natural numbers fails to be compact
- The compactness of $\mathbb{N}$ amounts to Bishop's LPO (Limited Principle of Omniscience).

More precisely, LPO is independent of MLTT

- False in realizability models (not computable)
  in topological models (not continuous)
- True in the model of classical sets (by choice)

Probably the simplest infinite example

$$\mathbb{N}_\infty := \Sigma \alpha : 2^{\mathbb{N}}, \Pi i : \mathbb{N}, \alpha_i \geq \alpha_{i+1}$$

That is, the type of decreasing binary sequences.

$$\underline{n} := 1^n 0^\omega$$

$$\infty := 1^\omega$$

We have an injection $\mathbb{N} \longrightarrow \mathbb{N}_\infty$

$$n \longmapsto \underline{n}$$

<u>Theorem</u> of HoTT/UF

The type $\mathbb{N}_\infty$ is compact.

(JSL '2013)

↳ Done in a weaker system (Gödel's system T)

Proof sketch (with the difficult part omitted)

- Given $p : \mathbb{N}_\infty \to 2$, (<u>not</u> assumed be continuous)

  define $\beta_n = \min(p\,\underline{0}, p\,\underline{1}, \dots, p\,\underline{n})$ Formula for the infimum of the set of roots.

- This is clearly decreasing.

- Now we check whether $p\beta = 0$ or $p\beta = 1$.

  (0) If $p\,\beta = 0$ then we've found a root.

  (1) If $p\,\beta = 1$ then $p\,\alpha = 1$ for all $\alpha : \mathbb{N}_\infty$ and so there is no root. (This is easy classically and less so constructively.)

In the pub $\mathbb{N}_\infty$ there is a person $\beta : \mathbb{N}_\infty$ such that if β drinks, then everybody drinks.

Some consequences (decision procedures)

(1) For every $p : \mathbb{N}_\infty \to 2$ either $\prod n : \mathbb{N}, p\,\underline{n} = 1$ or $\neg \prod n : \mathbb{N}, p\,\underline{n} = 1$

(JSL'2013)

quantification over the natural numbers! Not over $\mathbb{N}_\infty$.

(2) Given $f : \mathbb{N}_\infty \to \mathbb{N}$, we can decide whether it is continuous or not.

(3) There is some discontinuous $f : \mathbb{N}_\infty \to \mathbb{N}$ iff WLPO holds

(Bishop's principle of Weak Limited omniscience, $\prod p : \mathbb{N} \to 2, (\prod n. p\,n = 1) + \neg(\prod p\,n = 1)$

which is also independent of MLTT)

(MSCS'2015)

Some applications of the compactness of $\mathbb{N}_\infty$

1. Pierre Pradic & Chad E. Brown. Arxiv '2019

   Cantor-Bernstein implies excluded middle

   arxiv 1904.09193

   (Also implemented in Coq.)

2. Dag Normann & William Tait. Springer '2017

   On the computability of the Fan Functional

   (They use the system T compactness of $\mathbb{N}_\infty$ to fill a gap in an unpublished but widely circulated 1958 manuscript by Tait.)

Compact sets in our type theory

(1) $\mathbb{0}$, $\mathbb{1}$ and $\mathbb{N}_\infty$ are compact.

(2) If X and Y are compact then so are $X+Y$ and $\overbrace{X \times Y}^{\text{Baby Tychonoff.}}$.

(3) If X is a compact set and A is a family of compact sets indexed by X, then its disjoint union $\sum_{x:X} A\,x$ is a compact set.

(4) If furthermore (a) we have a function that picks an element of $A\,x$ for any given $x : X$, and
(b) the set X has at most one element,
then the cartesian product $\prod_{x:X} A\,x$ is compact. Micro-Tychonoff.

Does arbitrary Tychonoff hold?

Is the Cantor type $\mathbb{N} \to 2$ (provably) compact in our type theory?

No.

- The compactness of $\mathbb{N} \to 2$ is <u>independent</u>.

True in the topological topos (notions of compactness coincide)

False in Hyland's effective topos (Kleene tree to blame)

(realizability topos over Kleene's K_1)

True in the Kleene-Vesley topos

(realizability over Kleene's K_2)

Perhaps amazingly, these two toposes have the same simple types.

(more precisely, the full subcategories on the objects that arise as the interpretation of the simple types are equivalent.)

Building more compact sets

- The compact sets that we have constructed so far are all well-ordered.

(1) $\mathbb{0}$

$\mathbb{1}$

$\mathbb{N}_\infty$

(2) $X+Y$

$X \times Y$

(3) $\Sigma\, x : X,\, A x$

lexicographic order

- But we can't get very high ordinals with just the above.
- This is what we address next.

Extending families of compact sets

General situation:

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, "A"'] & & Y \arrow[dl, dotted, "B"] \\ & \mathcal{U} & \end{tikzcd}$$

j embedding

A type family

$\mathcal{U}$ type universe

B extension of A along j

Interested in:

$$\begin{tikzcd} \mathbb{N} \arrow[rr, hook, "\text{canonical embedding}"] \arrow[dr, "A"'] & & \mathbb{N}_\infty \arrow[dl, dotted, "B"] \\ & \mathcal{U} & \end{tikzcd}$$

Want: If $A\,x$ compact for every $x : X$, then $B\,y$ compact for every $y : Y$.

Because then: By (3), if Y is also compact, then $\Sigma\, y : Y,\ B\,y$ compact too.

$j'(y) = \sum_{x:X} jx = y$

Family extension problem

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d] & Y \arrow[dl, dotted] \\ \mathcal{U} & \end{tikzcd}$$

A B

(MSCS'2021. "Injective types in univalent mathematics".)

This set has at most one element. (because j is an embedding)

Smallest solution (left kan extension): $B\,y := \sum_{(x,-):j^{-1}(y)} A\,x$

Largest solution (right kan extension): $B\,y := \prod_{(x,-):j^{-1}(y)} A\,x$

It is this that works for the wish of the previous board.

why? By Micro-Tychonoff

Summary of the previous reasoning

$$\begin{tikzcd} X \arrow[r, hook, "j \text{ given}"] \arrow[d, "A \text{ given}"'] & Y \arrow[dl, dotted] \\ \mathcal{U} & \end{tikzcd}$$

$$B\,y := \prod_{x : j^{-1}(y)} A\,x$$

Special case of interest:

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] \arrow[d, "A"'] & \mathbb{N}_\infty \arrow[dl, dotted, "B"] \\ \mathcal{U} & \end{tikzcd}$$

Theorem If the set $A\,x$ is compact for every $x : X$, then then set $B\,y$ is compact for every $y : Y$.

Corollary If additionally Y is compact, then so is $\sum_{y : Y} B\,y$.

$\boxed{\text{In the special case of interest we have } B(\infty) \simeq \mathbb{1}}$

More

$$\begin{tikzcd} \mathbb{N} \arrow[r, hook, "j"] \arrow[dr, "A"'] & \mathbb{N}_\infty \arrow[d, dotted] \\ & \mathcal{U} \end{tikzcd}$$

$$B_y = \prod_{x : j^{-1}(y)} A_x .$$

Classically

This is a bijection (with noncomputable inverse)

Constructively

This is an injection whose image has empty complement.

$$\left(\sum_{x : \mathbb{N}} A_x \right) + 1 \hookrightarrow \sum_{y : \mathbb{N}_\infty} B_y$$

adds "isolated" point

adds point "at infinity".

Notation:

$$\Sigma_1 \; A_x \quad x : X$$

Notation:

$$\Sigma^1 \; A_x \quad x : X$$

what is the point of the previous discussion?

- The well-ordered set $\left(\sum_{x:\mathbb{N}} A x\right) + 1$ is not compact in general, even if $A x$ is compact for every $x : \mathbb{N}$.

- However, the (classically isomorphic) set $\sum_{y:\mathbb{N}_\infty} B y$ is compact.

$$\sum_{x:\mathbb{N}} {}_1\, A x \hookrightarrow \sum_{x:\mathbb{N}}{}^1\, A x$$

constructively, this embedding has empty complement.

Ordinal expressions OE

Inductively defined (a W type)

We can get much higher than ε_0 (c.f. Anton Setzer's work)

$$\mathrm{One} : OE$$
$$\mathrm{Add} : OE \to OE \to OE$$
$$\mathrm{Mul} : OE \to OE \to OE$$
$$\mathrm{Sum1} : (\mathbb{N} \to OE) \to OE$$

Two interpretations

$$[\![\mathrm{one}]\!]_1 = \mathbb{1}$$
$$[\![\mathrm{Add}\ e\ e']\!]_1 = [\![e]\!]_1 + [\![e']\!]_1$$
$$[\![\mathrm{Mul}\ e\ e']\!]_1 = [\![e]\!]_1 \times [\![e']\!]_1$$
$$[\![\mathrm{Sum1}\ e]\!]_1 = \underset{n:\mathbb{N}}{\Sigma_1} [\![e]\!]_1\, n$$

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$$[\![\mathrm{Sum1}\ e]\!]^1 = \underset{n:\mathbb{N}}{\Sigma^1} [\![e]\!]^1\, n$$

only difference

Theorems

The ordinal $[\![e]\!]_1$

- is discrete
- is a retract of $\mathbb{N}$
- so countable
- Not compact unless LPO holds

The ordinal $[\![e]\!]^1$

- is compact
- is a retract of $\mathbb{N} \to 2$
- so totally separated
- Is not countable unless LPO holds
- Is not discrete unless LPO holds

Even better:
Every decidable subset is either empty or has a least element.

- There is an order-preserving-reflecting embedding

$$[\![e]\!]_1 \hookrightarrow [\![e]\!]^1$$

whose image has empty complement.

- LPO $\Rightarrow$ this embedding is a bijection $\Rightarrow$ WLPO.

In models:
The embedding doesn't have a computable/continuous inverse.

Illustration | The ordinal $\omega+1$.

$$n \longmapsto 1^n 0^\omega$$

$$\mathbb{N} + \mathbb{1} \hookrightarrow \mathbb{N}_\infty$$

$$* \longmapsto 1^\omega$$

- Discrete
- compact iff LPO
- Countable

- Compact
- discrete iff WLPO
- countable iff LPO

- bijection iff LPO,
- but its image has empty complement.

Every decreasing sequence is of one of the forms $1^n 0^\omega$ and 1^ω.

There is no decreasing sequence other than $1^n 0$ and 1^ω.