Logico-Pluralistic Exploration of Foundational Theories with Computers

Christoph Benzmüller
U Bamberg & FU Berlin

jww colleagues, students & in particular: Dana Scott

"If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis."

(Leibniz, 1677)

Topos Institute Colloquium, June 16, 2022

Talk Structure

Research Motivation, Methodology & Framework

- AI and Representing Objects
- Logico-Pluralistic KR&R Methodology: LogiKEy
- Universal (Meta-)Logical Reasoning

Study/Exploration of Foundational Theories

- Axiomatization of Category Theory (Free Logic) … with …
- Further Foundational Studies
    - Metaphysics, Ethics & Law, …

Dana Scott

Conclusion

Artificial Intelligence (AI)

Weak AI: … solve specific problems
Strong AI: … everything humans can do (and possibly way more)

Strong AI requires at least (own working hypothesis):

1. Problem Solving (in the sense of weak AI)
2. Exploring and Acting in Unknown Territory (e.g. Mars-Rover)
3. Abstract & Rational Reasoning (e.g. exploration of a new theory in maths)
4. Self-Reflection (e.g. detect own mistakes, questioning the results of one's own thinking)
5. Social Interaction (e.g. adjusting personal goals and values to those of a community)

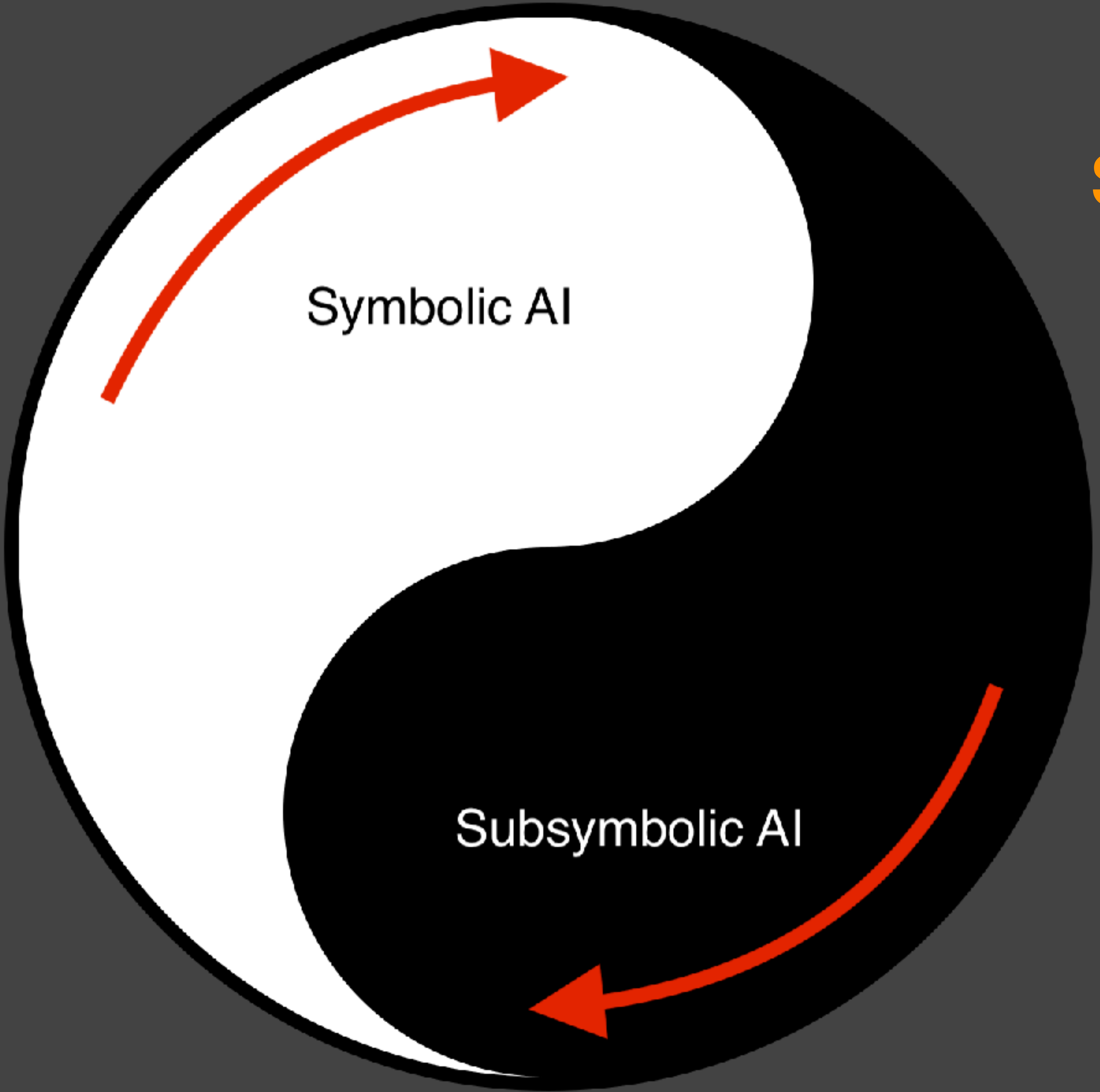
Machine Learning

Hybrid AI

Artificial Intelligence

symbolic

Mind
Reasoning
Deductive
Little Data
Causalities
Precise



subsymbolic

Brain
Learning
Inductive
Massive Data
Correlations
Robust

AI & Representing Objects

Abstract representations: objects of study in AI from the very beginning. Bibel calls them Representing Objects (ROBs).

- ROBs are key to symbolic AI
- ROBs can be experimenented with in the computer
- ROBs thus become physical/accessible (part of nature?)

Symbolic AI & logic in combination with computer experimentation on ROBs deserve increased attention as experimental (natural?) science.

Examples: mathematical theories, metaphysical theories, legal theories, etc., including even their **underlying logics**

Wolfgang Bibel
(German AI Pioneer)

SpringerLink

HAUPTBEITRAG | Open Access | Published: 19 May 2022

Komputer kreiert Wissenschaft

Wolfgang Bibel

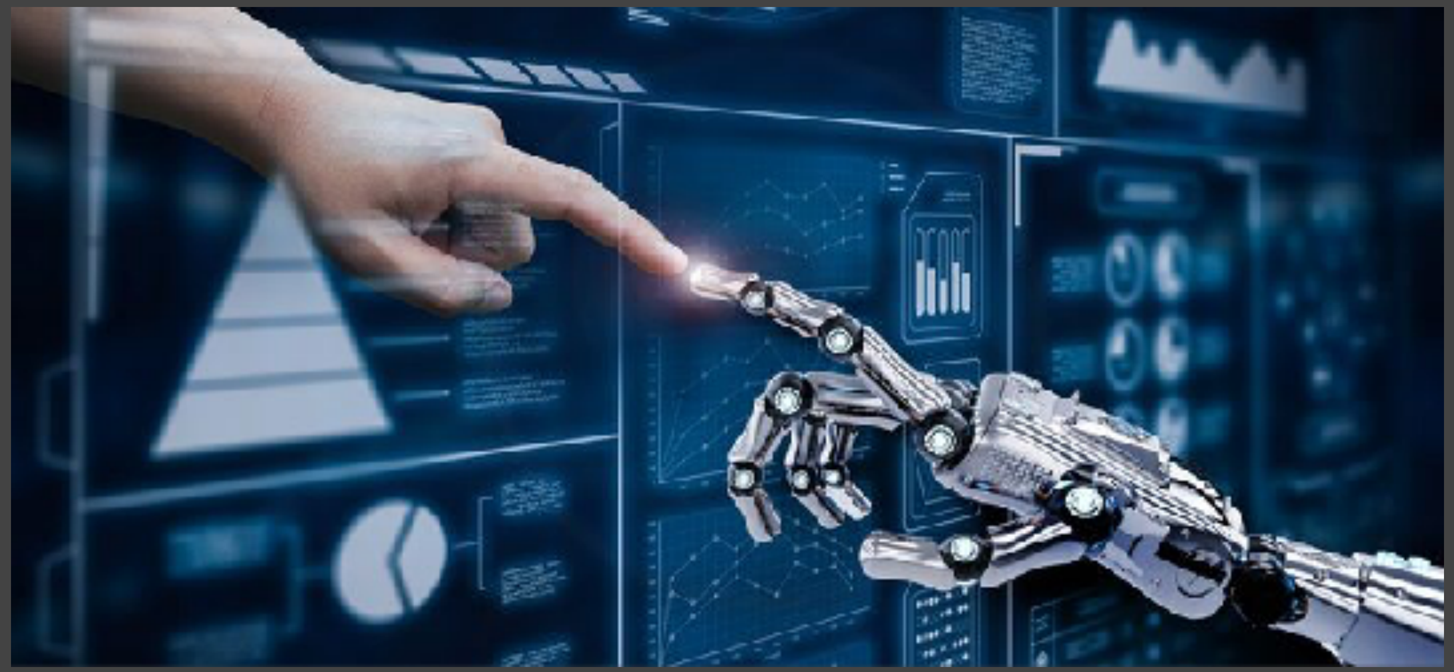
informatik Spektrum (2022) | Cite this article

102 Accesses | Metrics

AI & Representing Objects

Human-Computer Interaction

Representing object (logical representation)



Argument/theory

Experiment

AI & Representing Objects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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

News and Fake News

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

MEDIA & CULTURE

Is God Real? Scientists 'Prove' His Existence With Godel's Theory And MacBooks

HOME / SCIENCE NEWS

Researchers say they used MacBook to prove Goedel's God theorem

God exists, say Apple fanboy scientists

With the help of just one MacBook, two Germans formalize a theorem that confirms the existence of God.

Fake news by award winning journalist *Chris Matyszczyk (c|net)*

Experiments (Reading):

ECAI 2013, IJCAI & KI 2016, KI 2017
(with Bruno Woltzenlogel-Paleo)

Bulletin of the Section of Logic
Volume 49/2 (2020), pp. 127–148
http://dx.doi.org/10.18778/0138-0680.2020.08

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Christoph Benzmüller, David Fuenmayor

COMPUTER-SUPPORTED ANALYSIS OF POSITIVE PROPERTIES, ULTRAFILTERS AND MODAL COLLAPSE IN VARIANTS OF GÖDEL's ONTOLOGICAL ARGUMENT

. . .

arXiv > cs > arXiv:2202.06264

Computer Science > Logic in Computer Science

[Submitted on 13 Feb 2022]

A Simplified Variant of Gödel's Ontological Argument

Christoph Benzmüller

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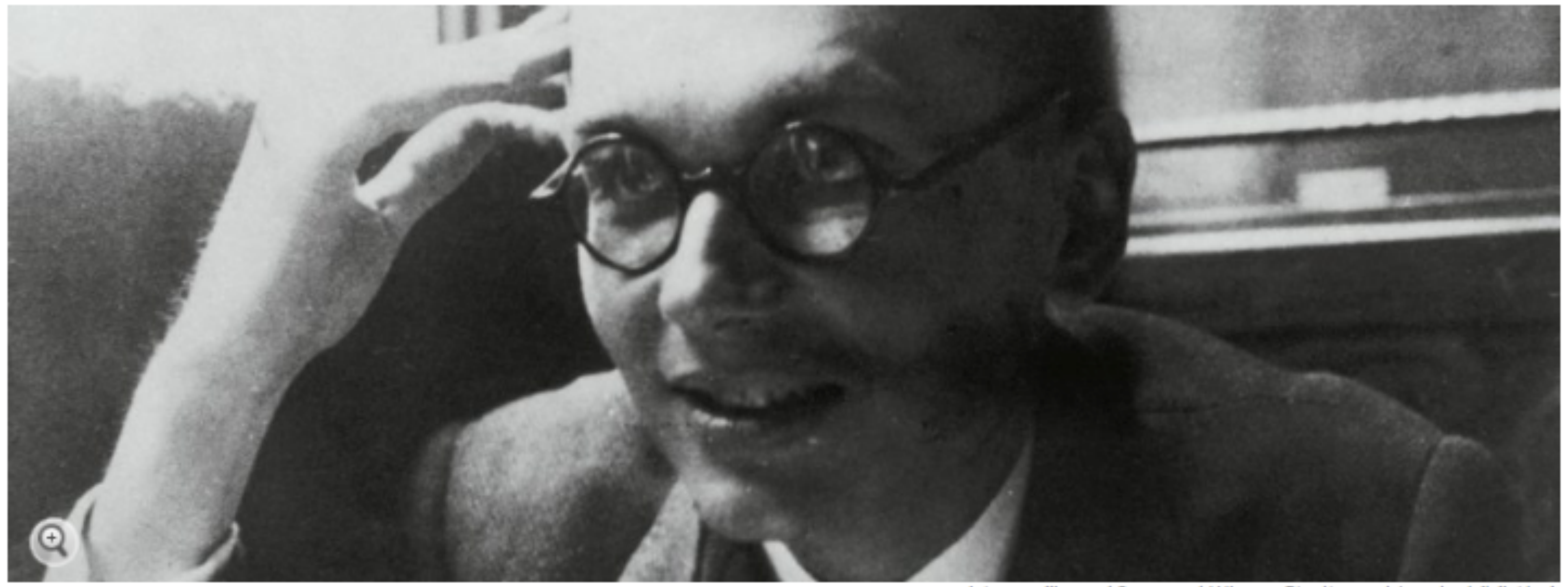

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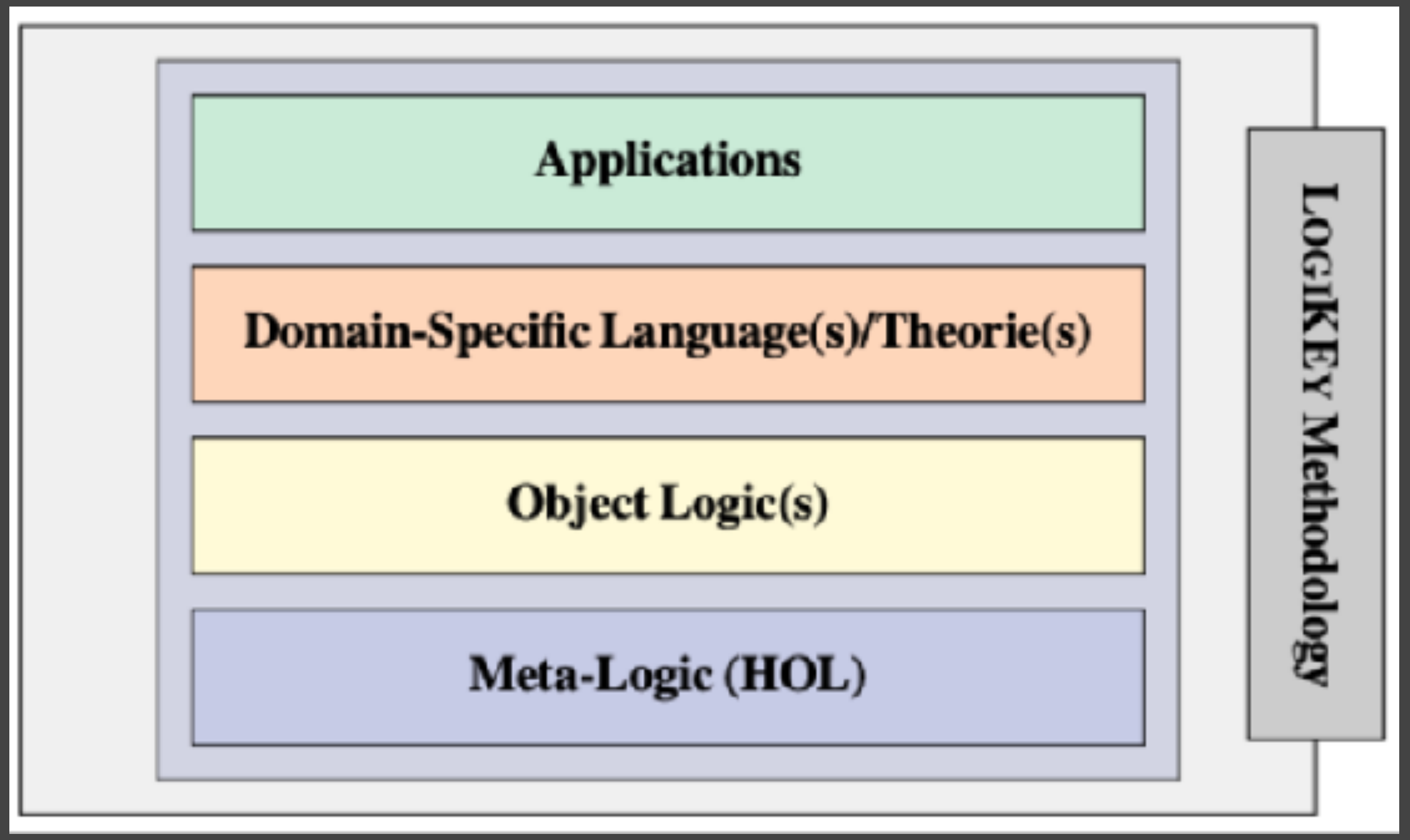
AI & Representing Objects

Human-Computer Interaction

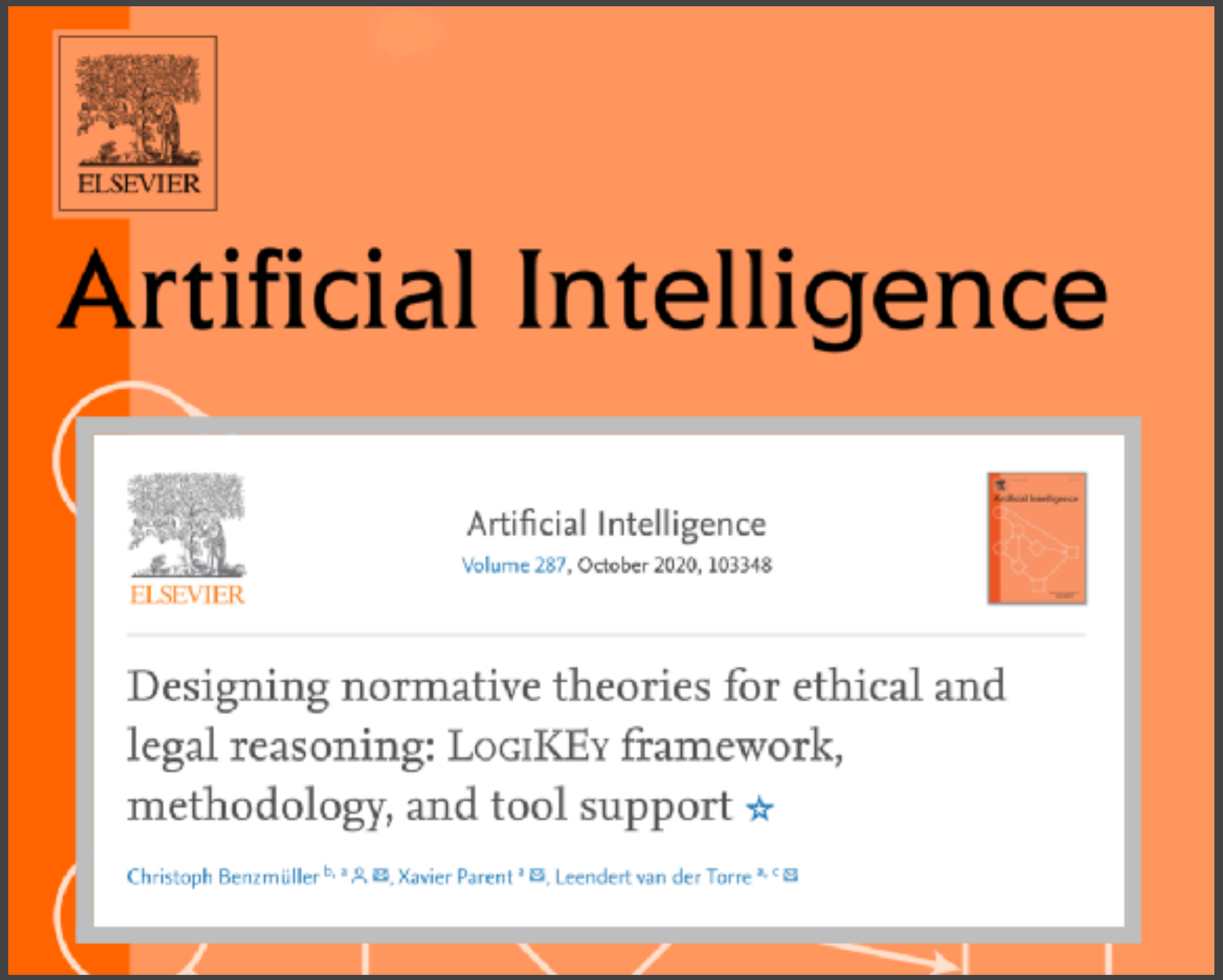
Representing object (logical representation)

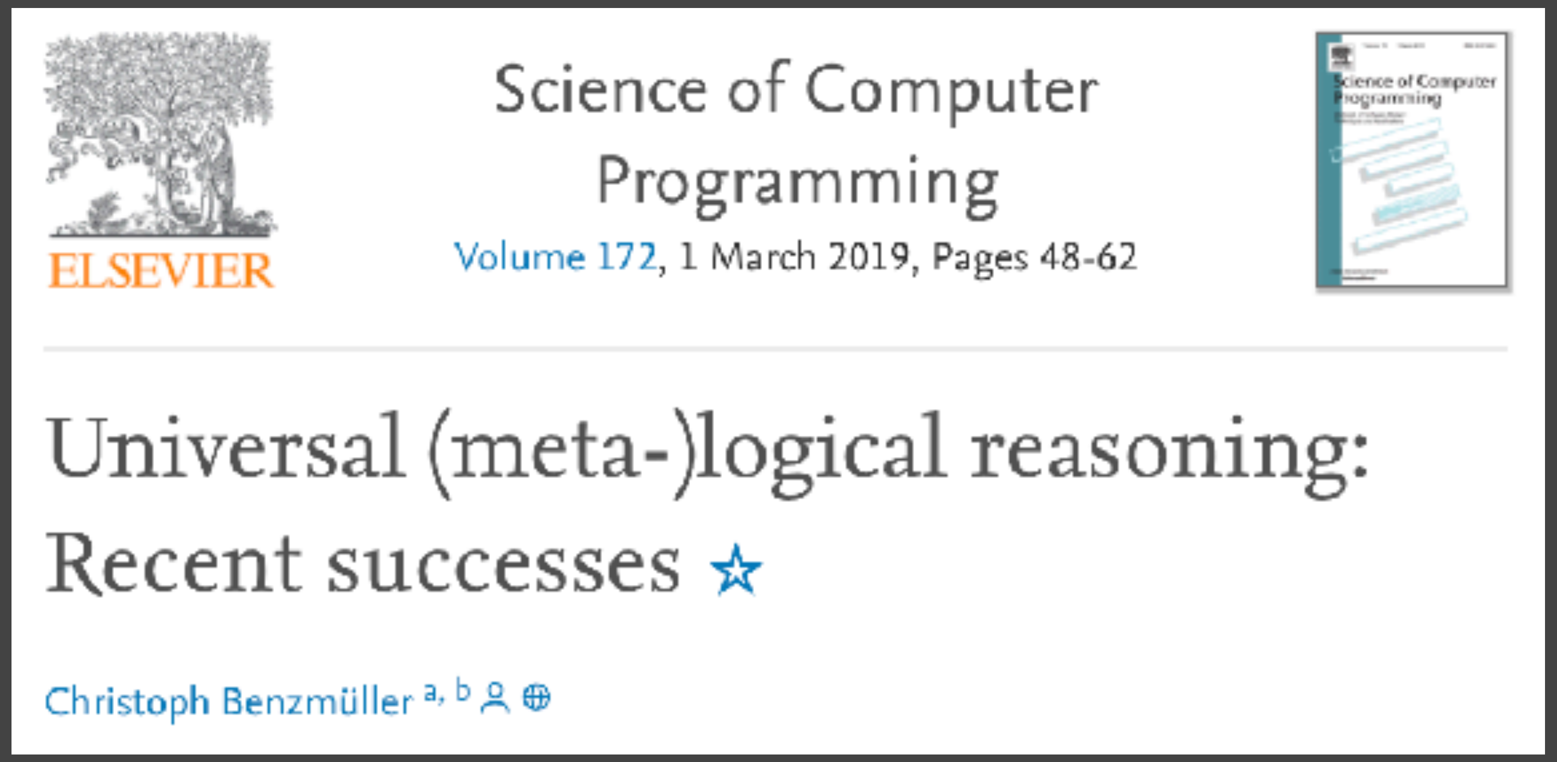


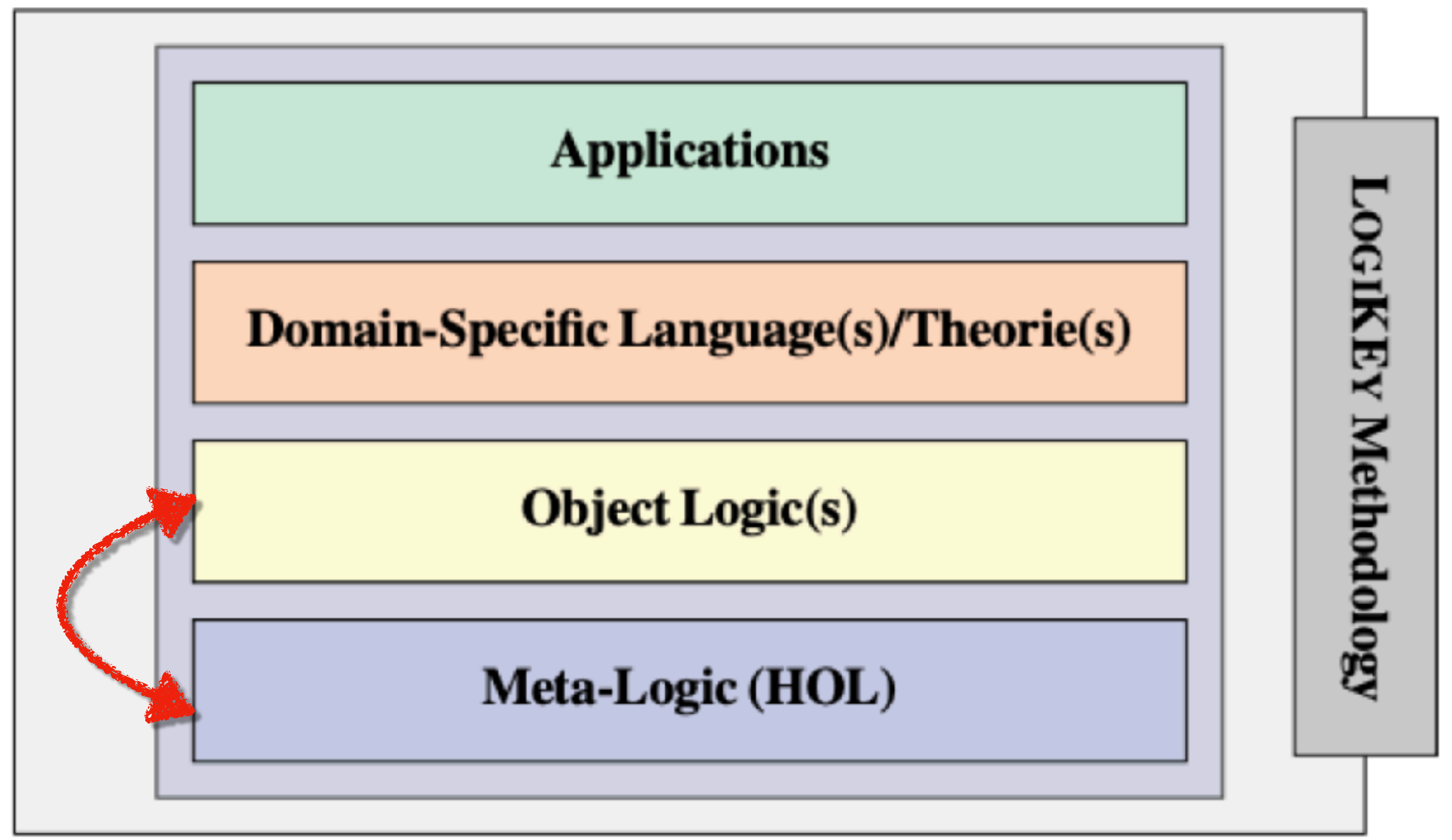
Argument/theory

Experiment

Universal (Meta-)Logical Reasoning







Different Layers — Flexibility (except HOL) — Parallel & Cyclic Development

Universal (Meta-)Logical Reasoning

Science of Computer Programming
Volume 172, 1 March 2019, Pages 48-62

Universal (meta-)logical reasoning: Recent successes ☆

Christoph Benzmüller [a, b]

…
developed from
…

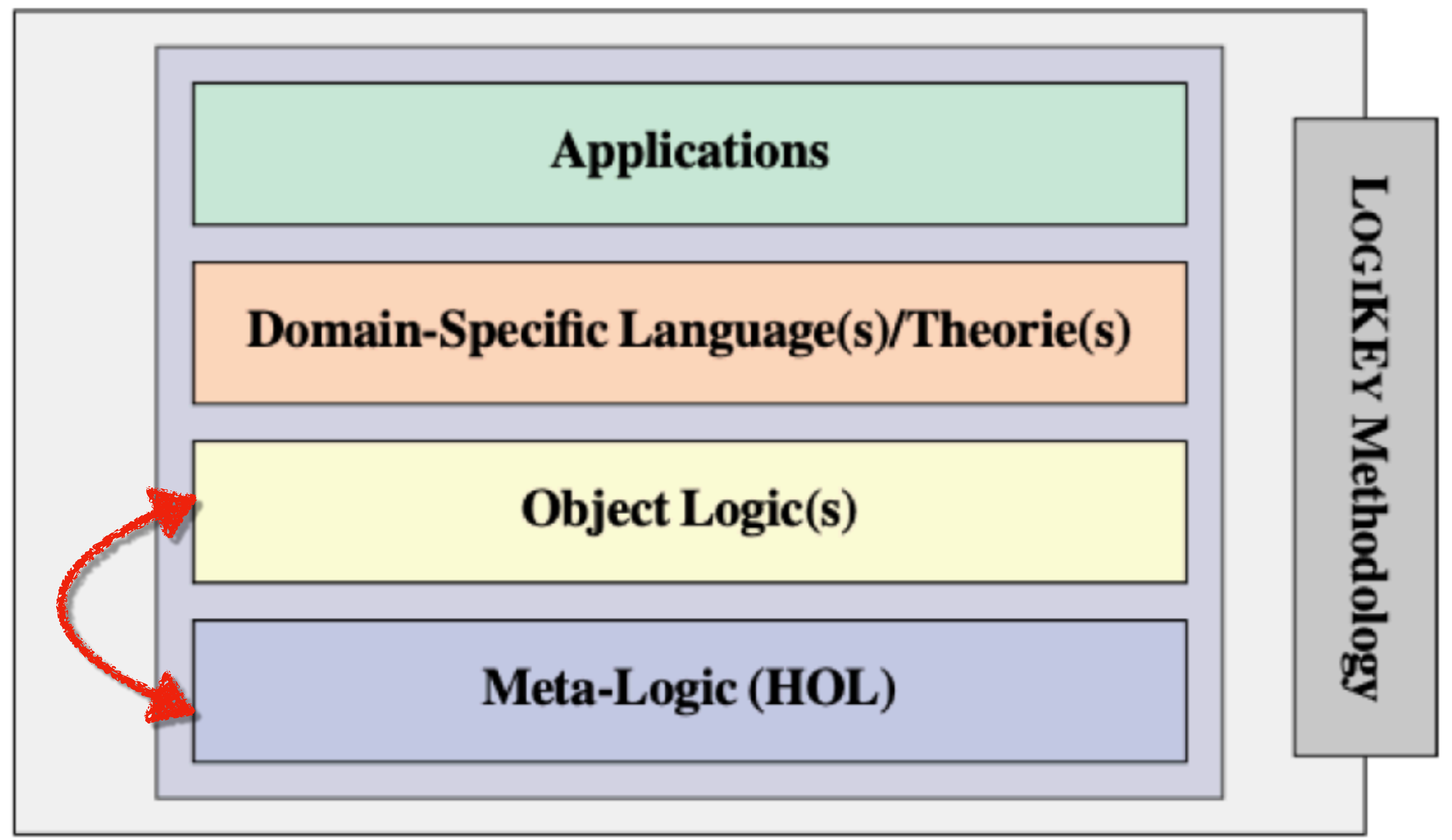
Springer Link

Published: 27 May 2012

Quantified Multimodal Logics in Simple Type Theory

Christoph Benzmüller & Lawrence C. Paulson

Logica Universalis 7, 7–20 (2013) | Cite this article



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Universal (Meta-)Logical Reasoning

Metaphysics

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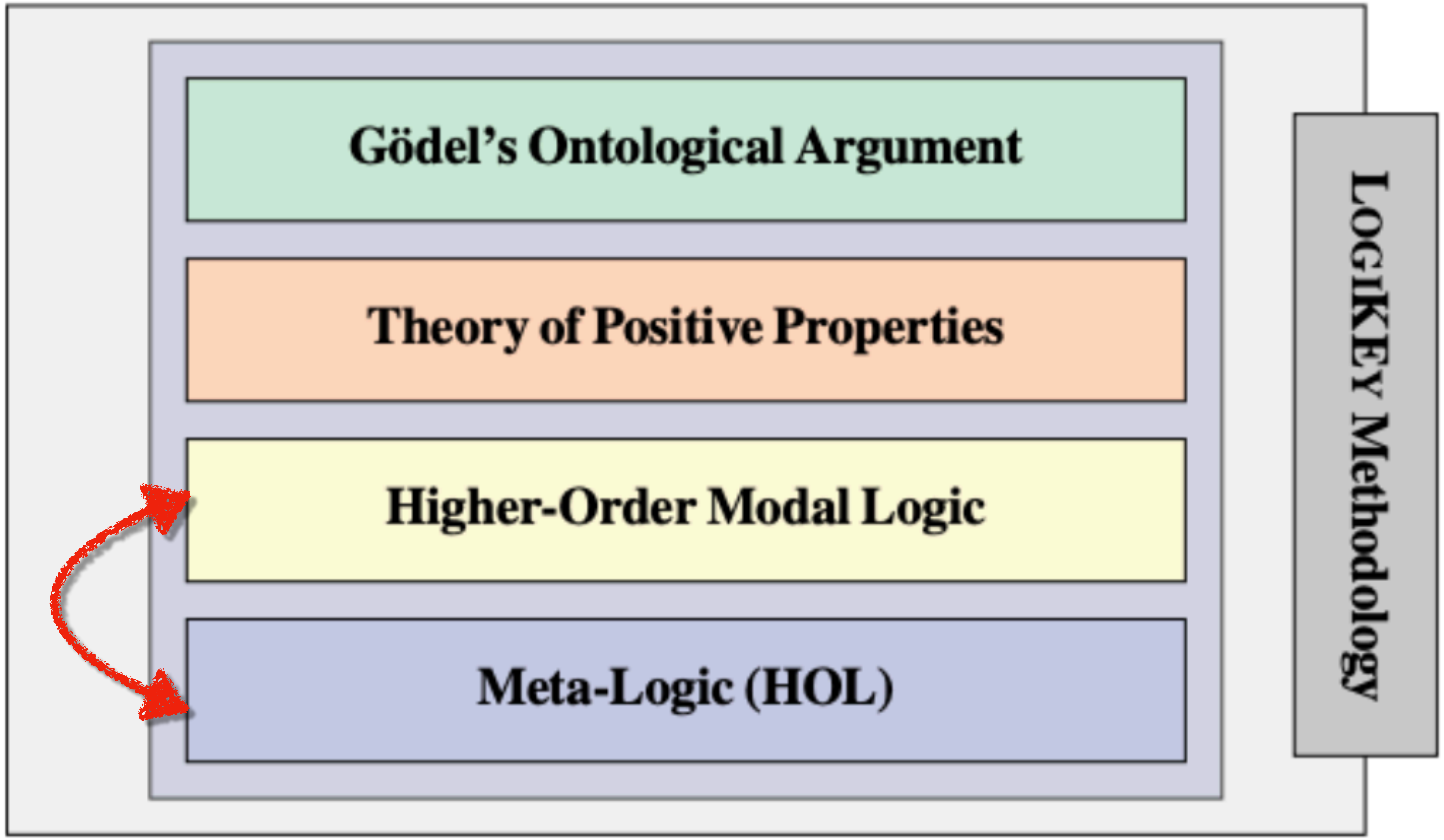
Automating Gödel's Ontological Proof of God's Existence with Higher-order Automated Theorem Provers

Authors	Christoph Benzmüller, Bruno Woltzenlogel Paleo
Pages	93 - 98
DOI	10.3233/978-1-61499-419-0-93
Series	Frontiers in Artificial Intelligence and Applications
Ebook	Volume 263: ECAI 2014



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Universal (Meta-)Logical Reasoning

Metaphysics

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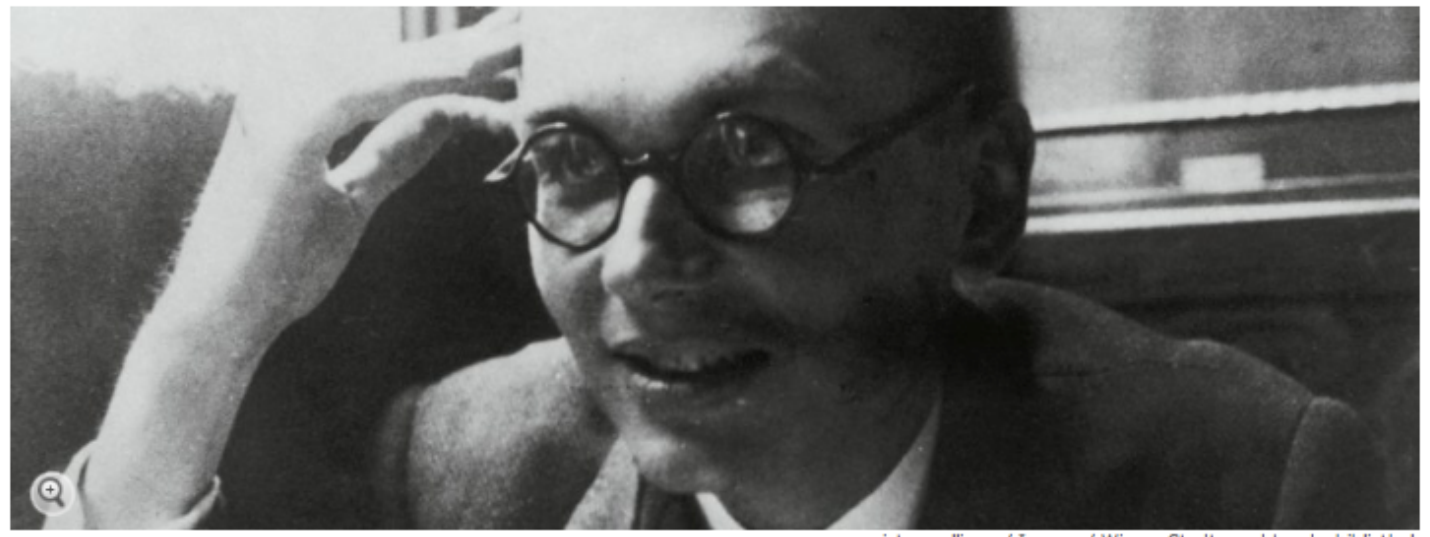

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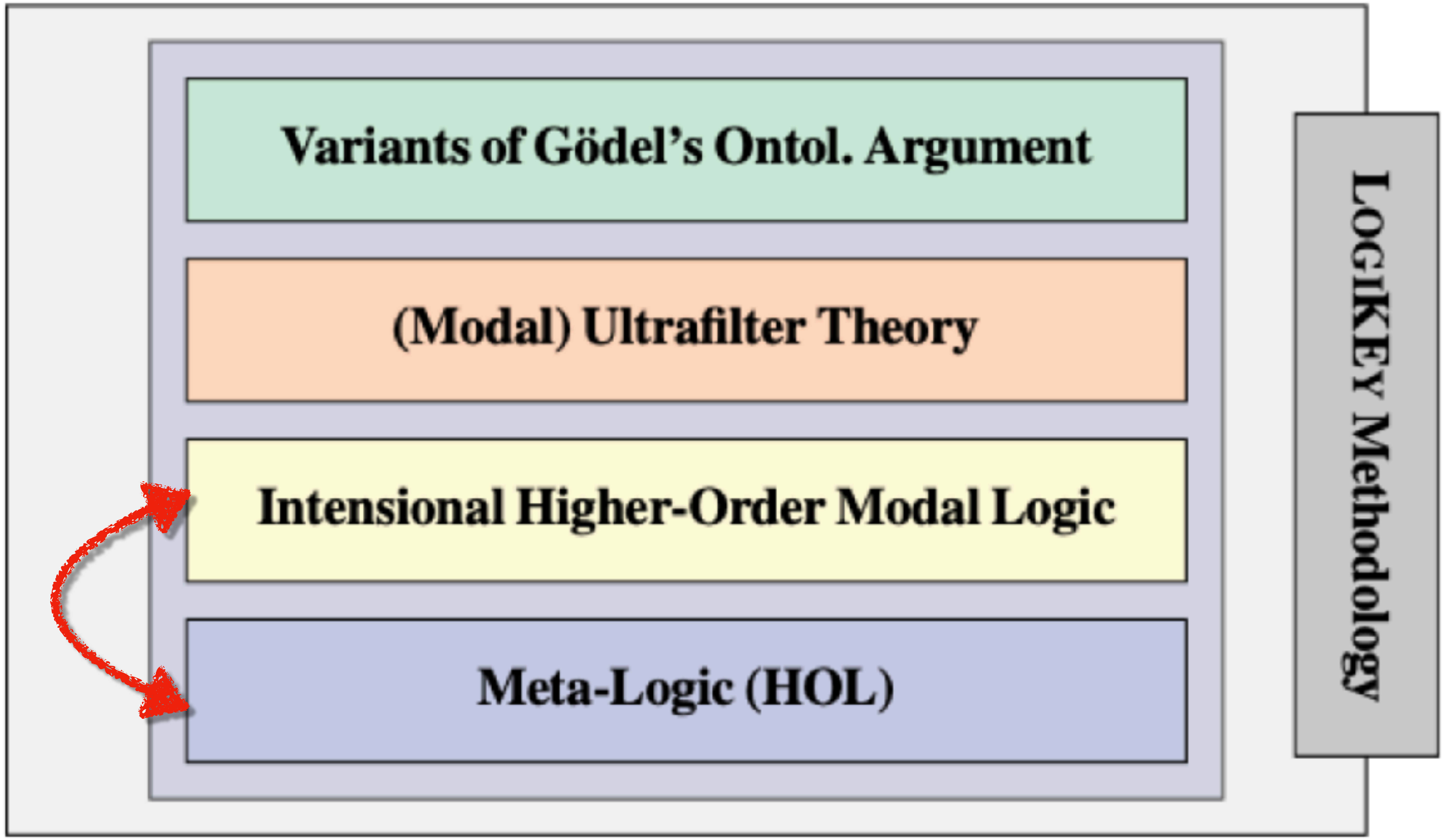

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Universal (Meta-)Logical Reasoning

Law & Ethics

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URN: urn:nbn:de:0030-drops-139028
URL: https://drops.dagstuhl.de/opus/volltexte/2021/13902/

Benzmüller, Christoph ; Fuenmayor, David

Value-Oriented Legal Argumentation in Isabelle/HOL

pdf-format: LIPIcs-ITP-2021-7.pdf (2 MB)

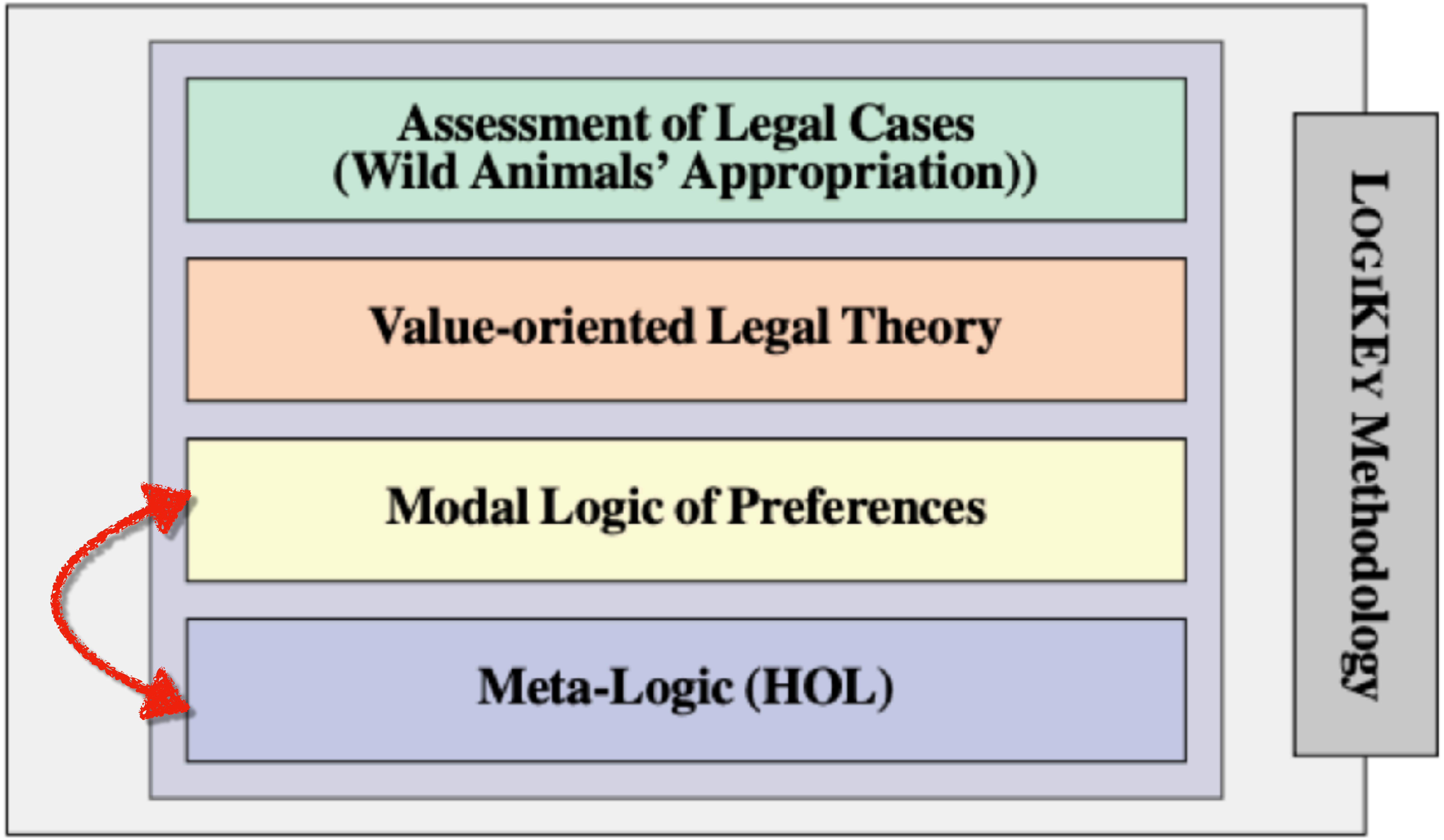
arXiv > cs > arXiv:2006.12789

Computer Science > Artificial Intelligence

[Submitted on 23 Jun 2020 (v1), last revised 30 Mar 2022 (this version, v5)]

Modelling Value-oriented Legal Reasoning in LogiKEy

Christoph Benzmüller, David Fuenmayor, Bertram Lomfeld



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Universal (Meta-)Logical Reasoning

Category Theory

International Congress on Mathematical Software
↳ ICMS 2016: **Mathematical Software – ICMS 2016** pp 43–50 | Cite as

Automating Free Logic in Isabelle/HOL

Christoph Benzmüller & Dana Scott

Springer Link

Published: 01 January 2019

Automating Free Logic in HOL, with an Experimental Application in Category Theory

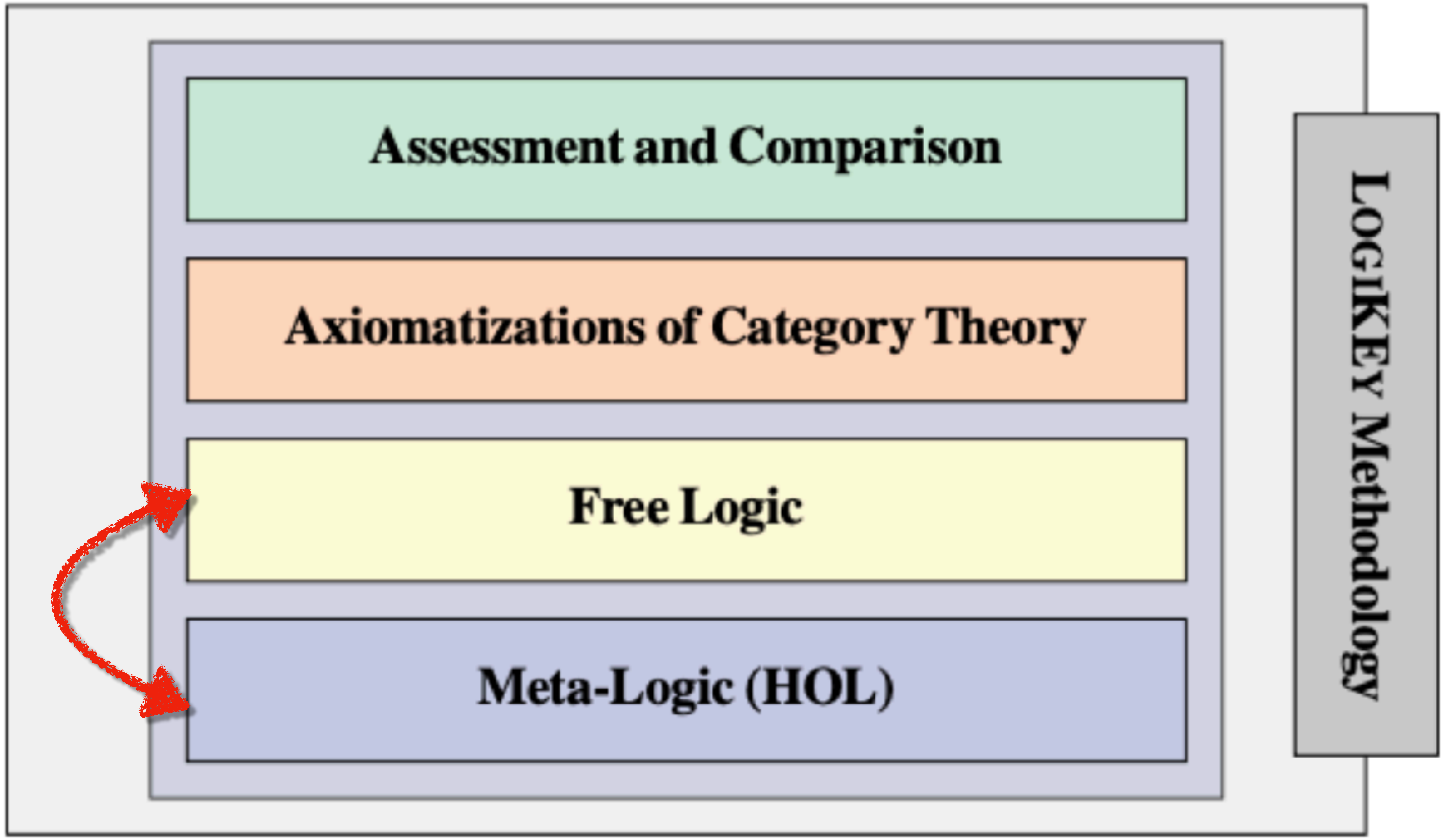
Christoph Benzmüller & Dana S. Scott

Journal of Automated Reasoning **64**, 53–72 (2020) | Cite this article

International Conference on Relational and Algebraic Methods in Computer Science
↳ RAMiCS 2020: **Relational and Algebraic Methods in Computer Science** pp 302–317

Computer-Supported Exploration of a Categorical Axiomatization of Modeloids

Lucca Tiemens, Dana S. Scott, Christoph Benzmüller & Miroslav Benda

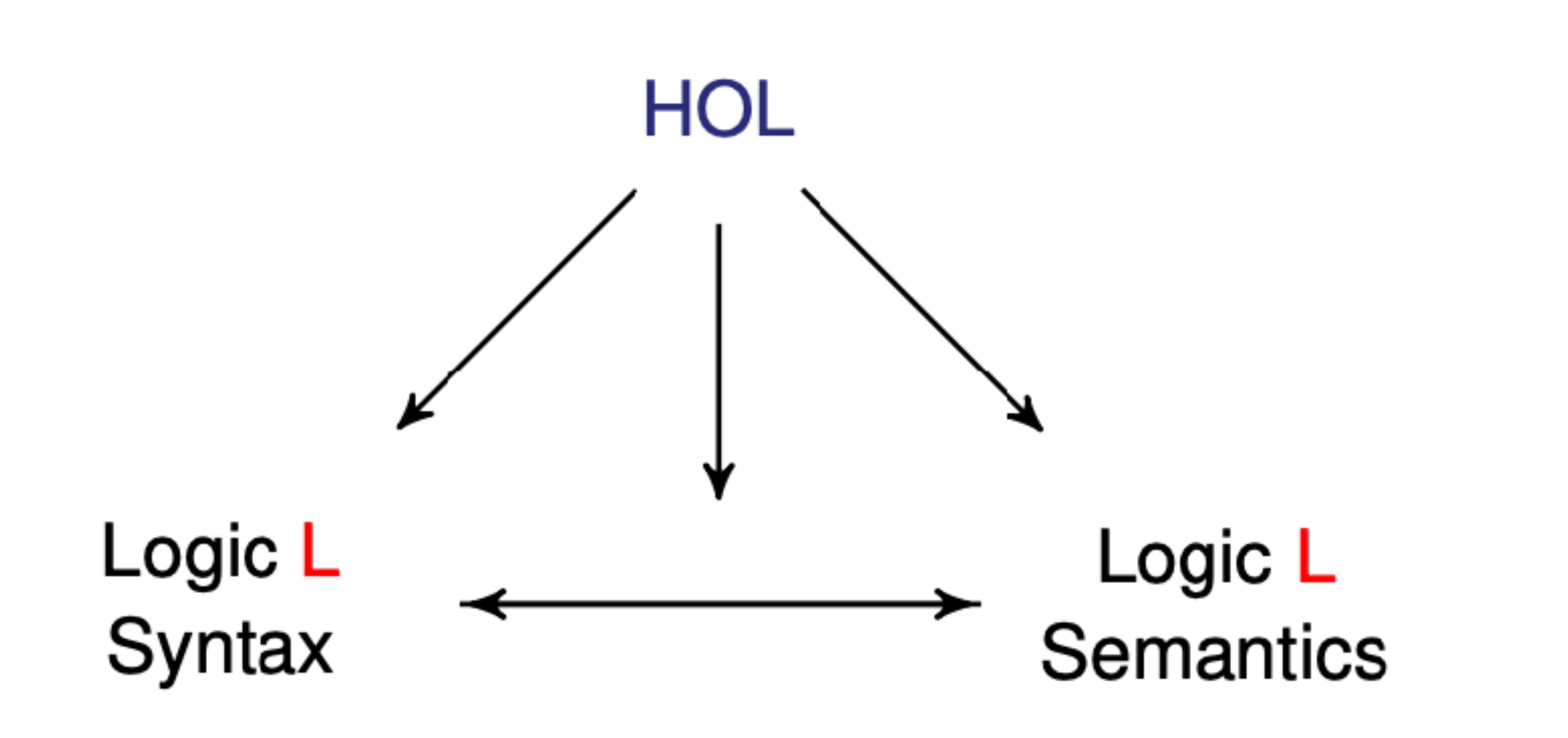


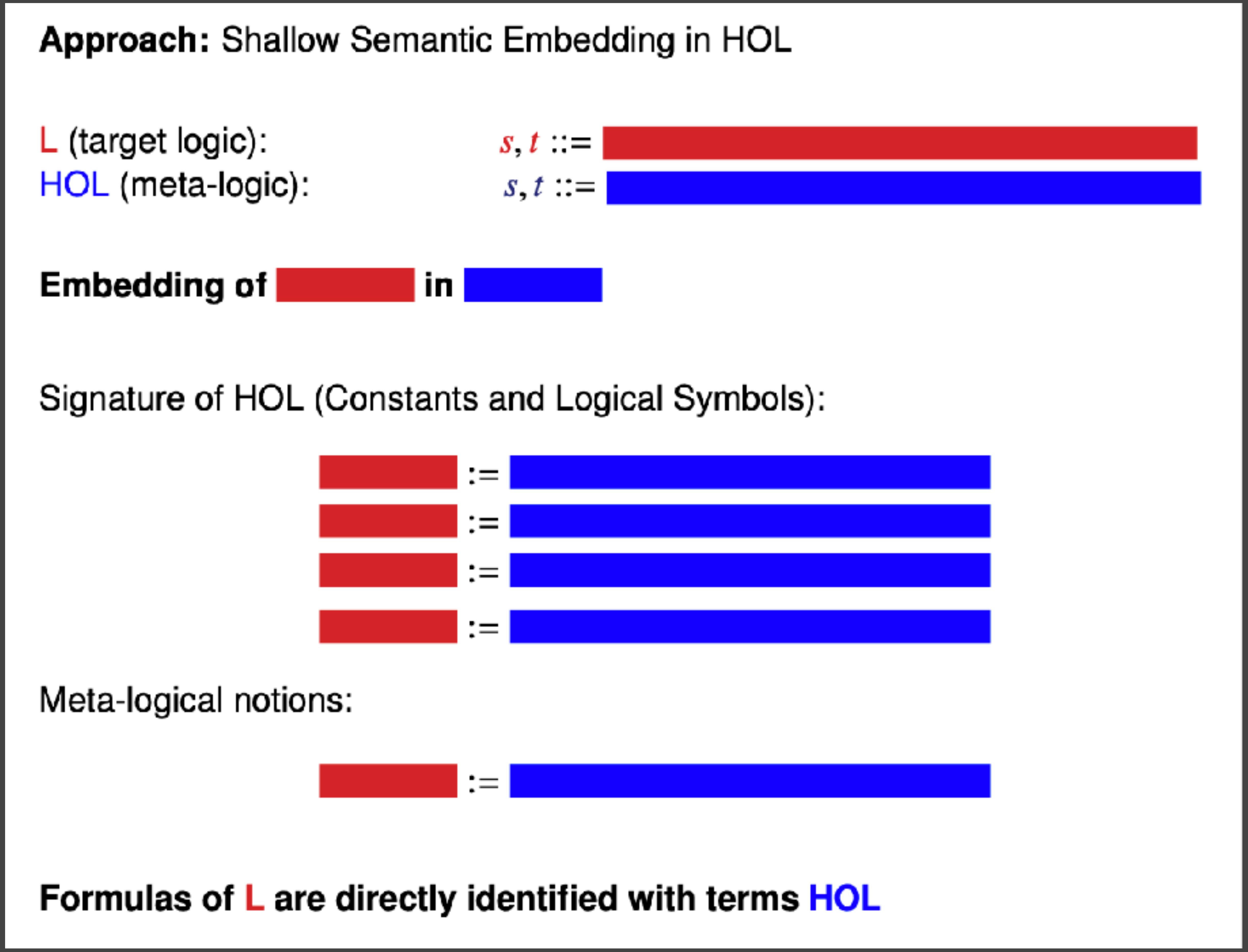
Different Layers — Flexibility (except HOL) — Parallel & Cyclic Development

Universal (Meta-)Logical Reasoning

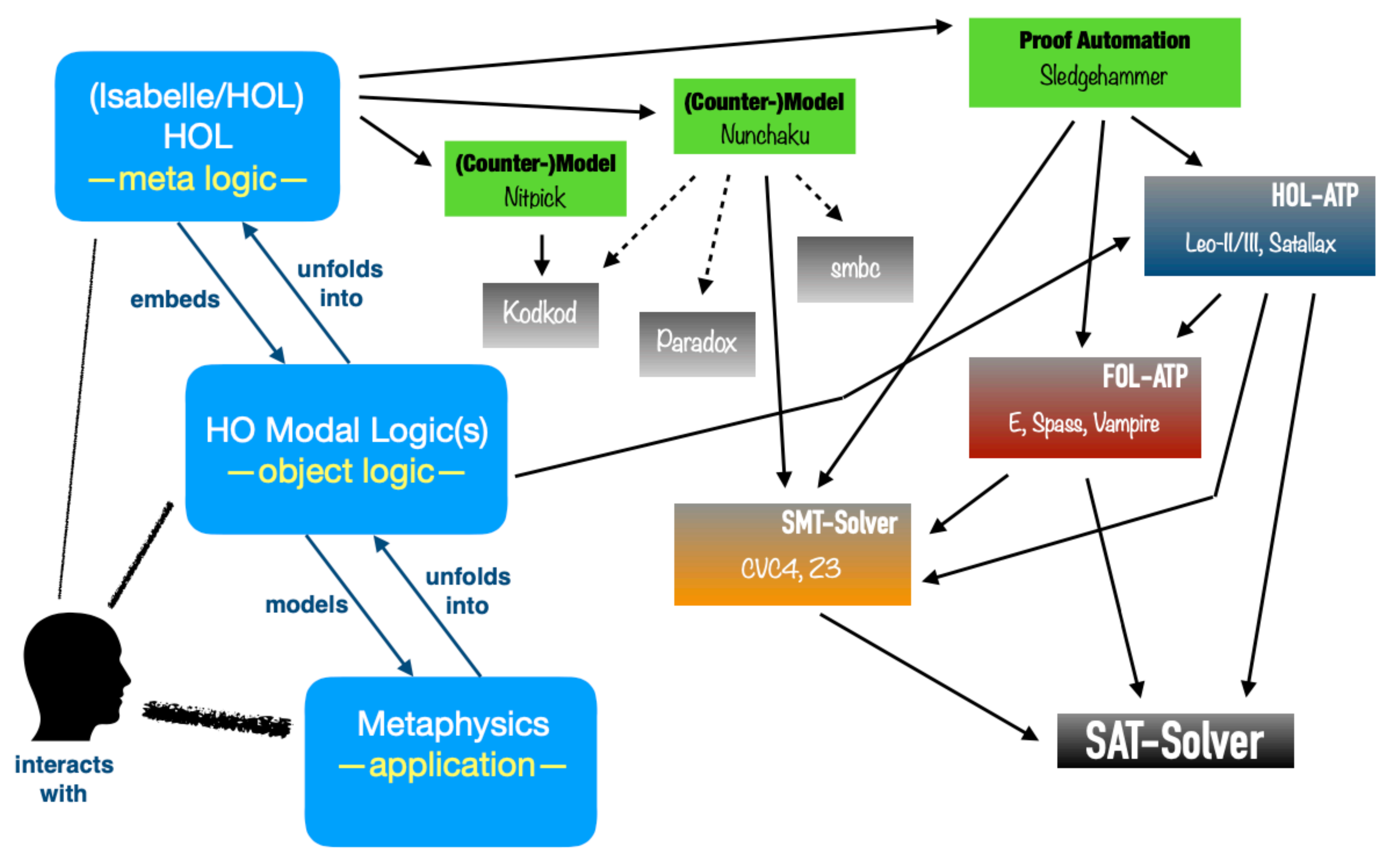
via Shallow Semantical Embeddings in Higher-Order Logic (HOL)

> “If we had it [a *characteristica universalis*], we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis.”
>
> (Leibniz, 1677)

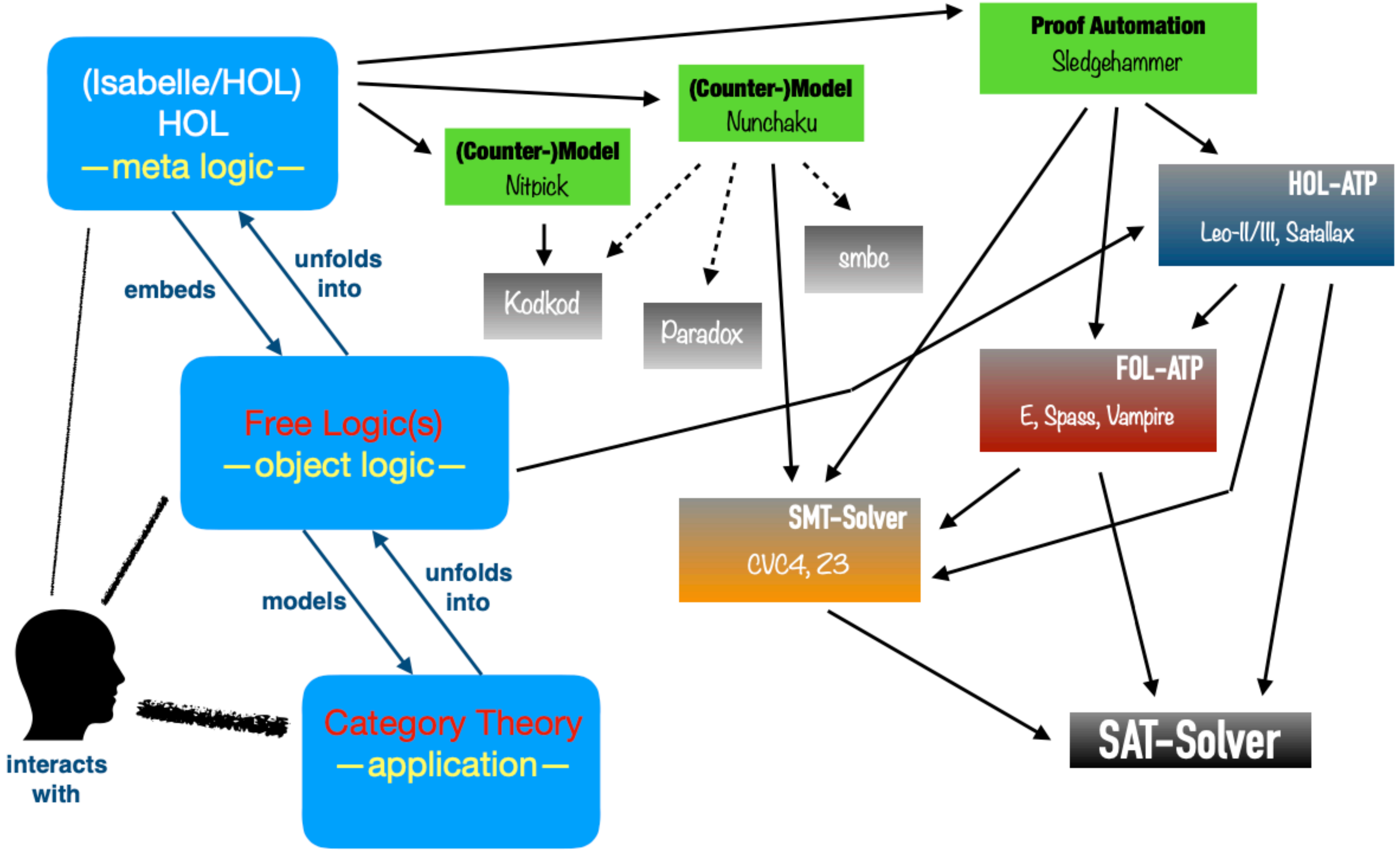




defining equations are passed to HOL theorem prover(s)



(Isabelle/HOL)
HOL
—meta logic—
(Counter-)Model
Nitpick
(Counter-)Model
Nunchaku
Proof Automation
Sledgehammer
HOL-ATP
Leo-II/III, Satallax
Kodkod
Paradox
smbc
embeds
unfolds into
HO Modal Logic(s)
—object logic—
FOL-ATP
E, Spass, Vampire
SMT-Solver
CVC4, Z3
models
unfolds into
interacts with
Metaphysics
—application—
SAT-Solver


(Isabelle/HOL)
HOL
—meta logic—
embeds
unfolds
into
Free Logic(s)
—object logic—
models
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into
Category Theory
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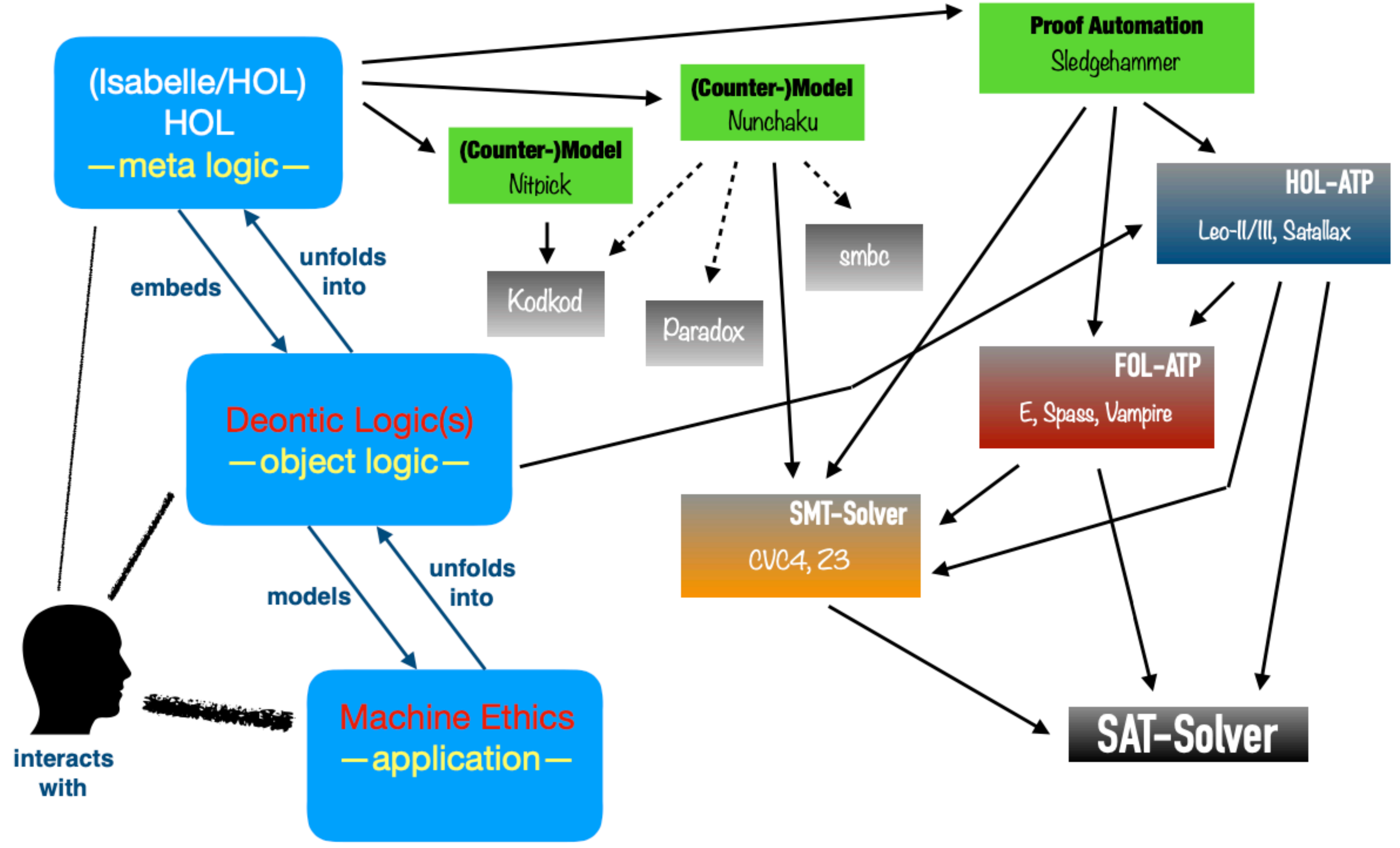
(Isabelle/HOL)
HOL
—meta logic—
embeds
unfolds into
Deontic Logic(s)
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models
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Machine Ethics
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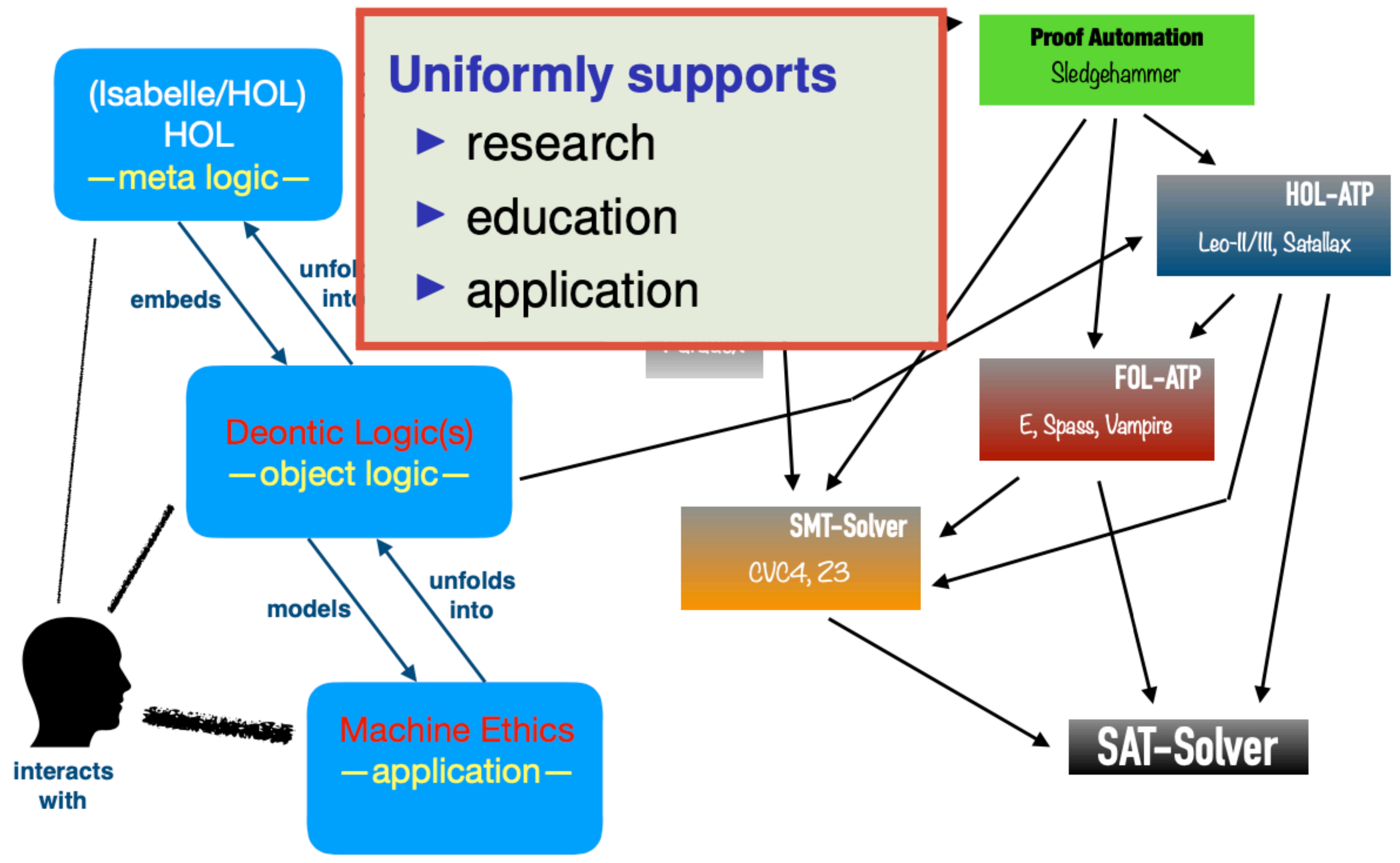

Uniformly supports
▶ research
▶ education
▶ application
(Isabelle/HOL)
HOL
—meta logic—
embeds
Deontic Logic(s)
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models
unfolds
into
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Ontological Argument — Results

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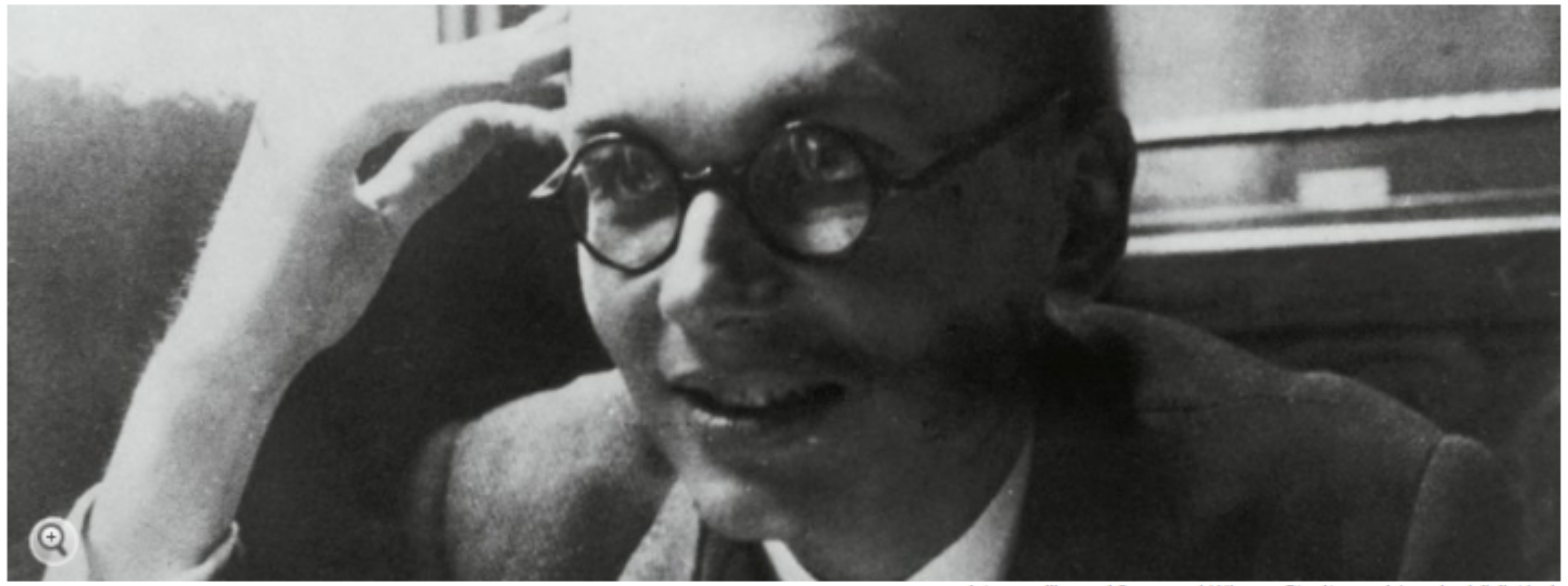

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Various insights not known before!

Study/Exploration of Foundational Theories

Axiom Systems for Category Theory

… explored in Free Logic … embedded in HOL

Axiomatization of Category Theory in Free Logic

Scott 1967

Dana Scott. "Existence and description in formal logic." In: Bertrand Russell: Philosopher of the Century, edited by R. Schoenman. George Allen & Unwin, London, 1967, pp. 181-200. Reprinted with additions in: Philosophical Application of Free Logic, edited by K. Lambert. Oxford Universitry Press, 1991, pp. 28 - 48.

16

DANA SCOTT

Existence and Description in Formal Logic

The problem of what to do with improper descriptive phrases has bothered logicians for a long time. There have been three major suggestions of how to treat descriptions usually associated with the names of Russell, Frege and Hilbert-Bernays. The author does not consider any of these approaches really satisfactory. In many ways Russell's idea is most attractive because of its simplicity. However, on s… of …

Technically the idea is to permit a wider interpretation of *free* variables. All bound variables retain their usual existential import (when we say something exists it does exist), but free variables behave in a more "schematic" way. Thus there will be no restrictions on the use of *modus ponens* or on the rule of *substitution* involving free variables and their occurrences. The laws of quantifiers require some modification, however, to make the existential assumptions explicit. The modification is very straightforward, and I shall argue that what has to be done is simply what is done naturally in making a *relativization* of quantifiers from a larger domain to a subdomain. Again in intuitionstic logic we have to take care over relativization, because we cannot say that either the subdomain is empty or not - thus a given element may be only "partially" in the subdomain.

Scott 1977

IDENTITY AND EXISTENCE IN INTUITIONISTIC LOGIC

Dana Scott

Merton College, Oxford, England

Standard formulations of intuitionistic … category theorists, generally do not take in… (For a recent reference see Makkai and Reyes… there is a simple psychological reason: we d… …ist. Certainly we shoul… …on explicit. In classic… …possible to split the d… …question does or does n… …s, and the circumstance … …ns, for example. Many p… …ocate in a mild way in this paper what I consider a simple …al formulation of logic allowing reference to partial elements. …be entirely formal here, but for the model theory of the system …nsult Fourman and Scott [10] for interpretations over a complete …this includes the so-called Kripke models) and Fourman [8] …en in 1975) for the interpretation in an arbitrary topos.

(1) $Ex \leftrightarrow Edom(x)$

(2) $Ex \leftrightarrow Ecod(x)$

(3) $E(x\circ y) \leftrightarrow dom(x) = cod(y)$

(4) $x\circ(y\circ z) \equiv (x\circ y)\circ z$

(5) $x\circ dom(x) \equiv x$

(6) $cod(x)\circ x \equiv x$

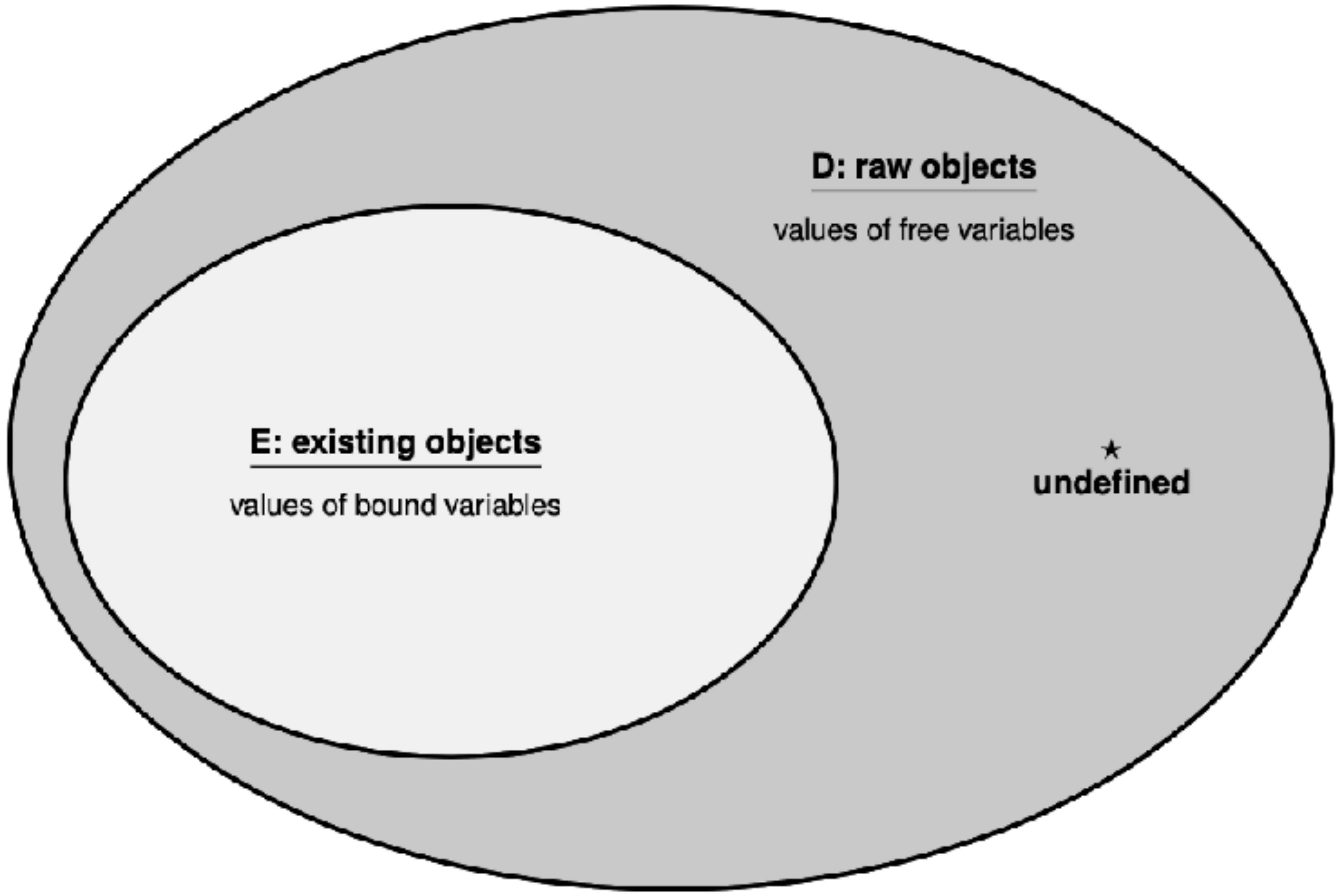
Free Logic

Existence and Description in Formal Logic (Dana Scott), 1967

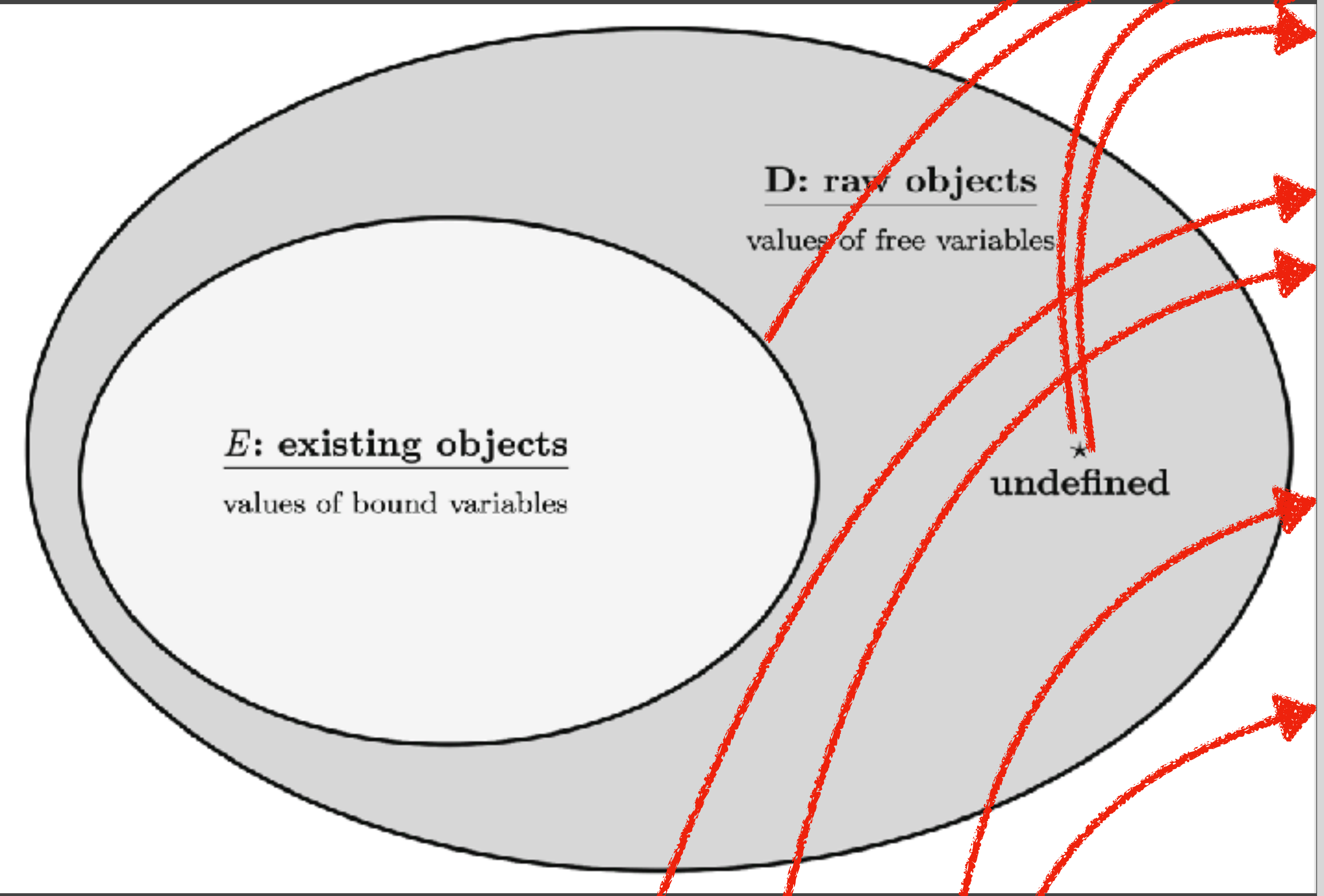
Principle 1: Bound individual variables range over domain $E \subset D$

Principle 2: Values of terms and free variables are in D, not necessarily in E only.

Principle 3: Domain E may be empty



Free Logic in HOL



- Free connectives ¬ and → … as in HOL
- Free quantifier ∀ relativized by E
- Free description constrained by E

```
theory AxiomaticCategoryTheory_FullFreeLogic imports Main
begin
typedecl i  (*Type for individuals*)
consts fExistence:: "i⇒bool" ("E") (*Existence/definedness predicate in free logic*)
consts fStar:: "i" ("⋆")   (*Distinguished symbol for undefinedness*)
axiomatization where fStarAxiom: "¬E(⋆)" (*⋆ is a ``non-existing'' object in D.*)

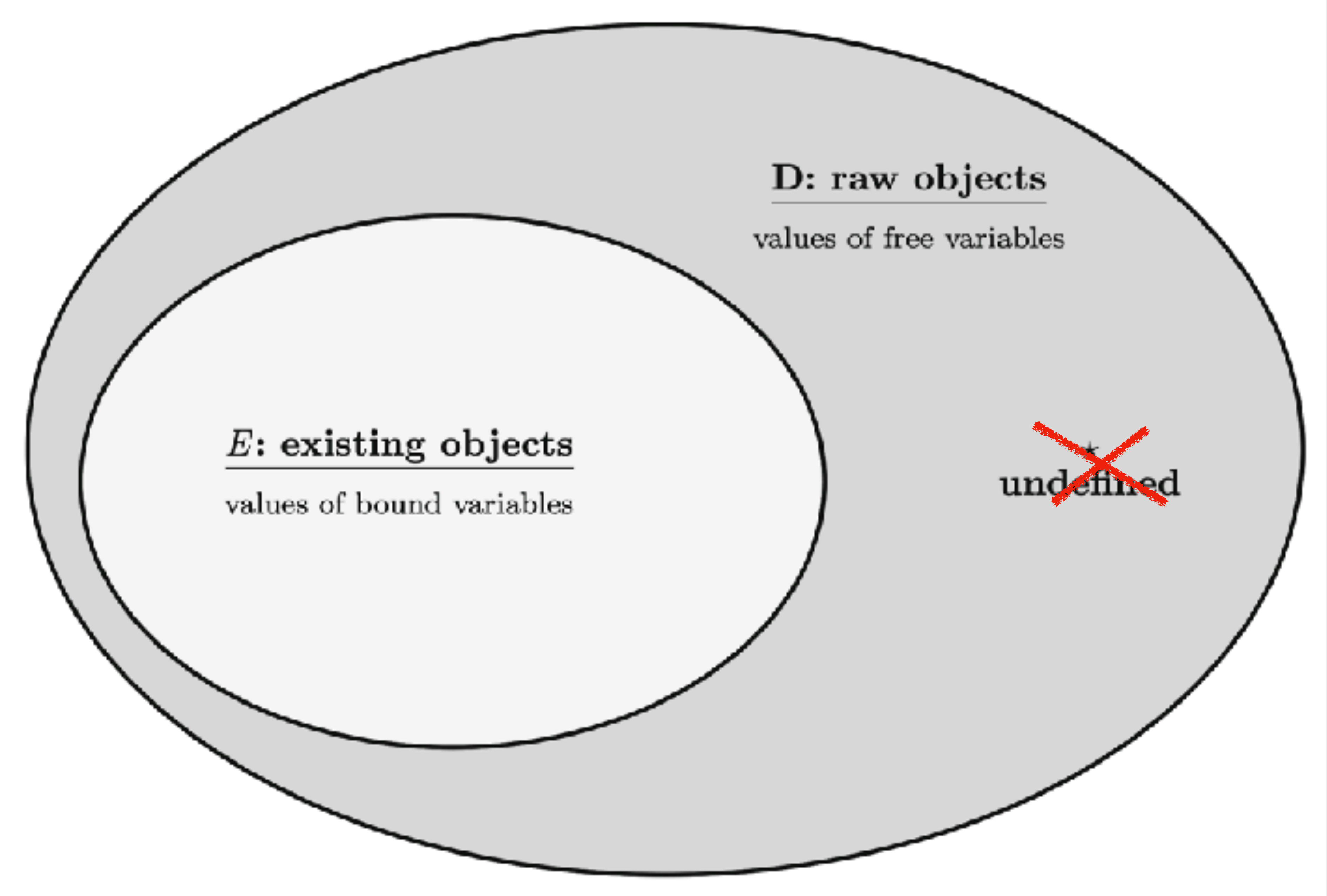
abbreviation fNot ("¬") (*Free negation*)
 where "¬φ ≡ ¬φ"
abbreviation fImplies (infixr "→" 13) (*Free implication*)
 where "φ → ψ ≡ φ ⟶ ψ"
abbreviation fIdentity (infixr "=" 13) (*Free identity*)
 where "l = r ≡ l = r"
abbreviation fForall ("∀") (*Free universal quantification guarded by @{text "E"}*)
 where "∀Φ ≡ ∀x. E x ⟶ Φ x"
abbreviation fForallBinder (binder "∀" [8] 9) (*Binder notation*)
 where "∀x. φ x ≡ ∀φ"
abbreviation fThat:: "(i⇒bool)⇒i" ("I")
 where "IΦ ≡ if ∃x. E(x) ∧ Φ(x) ∧ (∀y. (E(y) ∧ Φ(y)) ⟶ (y = x))
              then THE x. E(x) ∧ Φ(x)
              else ⋆"
abbreviation fThatBinder:: "(i⇒bool)⇒i"  (binder "I" [8] 9)
 where "Ix. φ(x) ≡ I(φ)"

text ‹ Further free logic connectives can now be defined as usual. ›

abbreviation fOr (infixr "∨" 11)
 where "φ ∨ ψ ≡ (¬φ) → ψ"
abbreviation fAnd (infixr "∧" 12)
 where "φ ∧ ψ ≡ ¬(¬φ ∨ ¬ψ)"
abbreviation fImplied (infixr "←" 13)
 where "φ ← ψ ≡ ψ → φ"
abbreviation fEquiv (infixr "↔" 15)
 where "φ ↔ ψ ≡ (φ → ψ) ∧ (ψ → φ)"
abbreviation fExists ("∃")
 where "∃Φ ≡ ¬(∀(λy. ¬(Φ y)))"
abbreviation fExistsBinder (binder "∃" [8]9)
 where "∃x. φ x ≡ ∃φ"
```

As usual

Free Logic in HOL

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abbreviation fForallBinder (binder "∀" [8] 9) (*Binder notation*)
 where "∀x. φ x ≡ ∀φ"
abbreviation fThat:: "(i⇒bool)⇒i" ("I")
 where "IΦ ≡ if ∃x. E(x) ∧ Φ(x) ∧ (∀y. (E(y) ∧ Φ(y)) ⟶ (y = x))
              then THE x. E(x) ∧ Φ(x)
              else ⋆"
abbreviation fThatBinder:: "(i⇒bool)⇒i"  (binder "I" [8] 9)
```

```
abbreviation fForall ("∀") (*Free universal quantification guarded by @{text "E"}*)
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              then THE x. E(x) ∧ Φ(x)
              else ⋆"
abbreviation fThatBinder:: "(i⇒bool)⇒i"  (binder "I" [8] 9)
 where "Ix. φ(x) ≡ I(φ)"
```

Axioms Set I

Generalized Monoids

Axioms Set II

Axioms Set III

Axioms Set IV

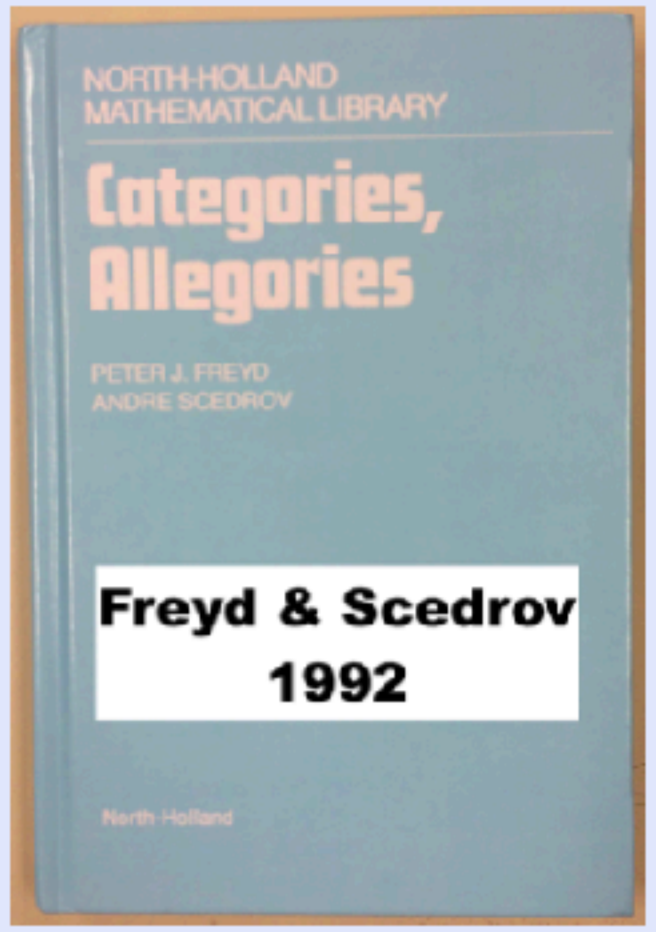
Axioms Set V

Dana Scott's Axioms from 1977

—?—

Axioms Set VI

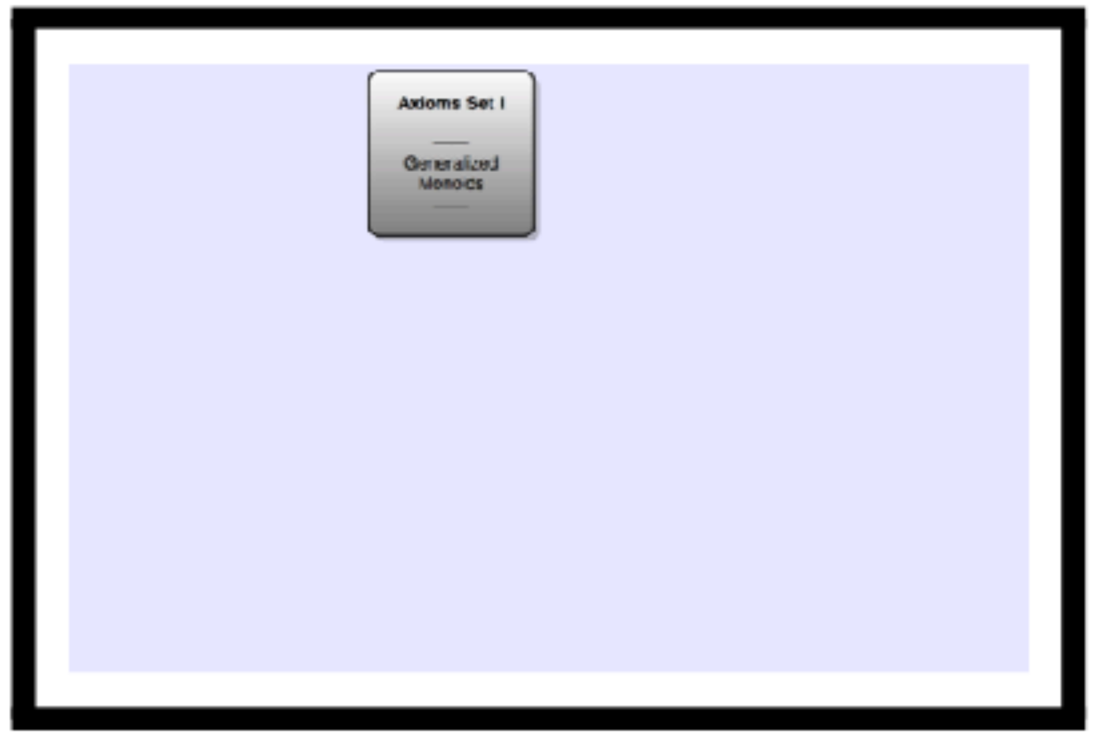
Freyd & Scedrov's Axioms from 1992



Dana Scott

all equivalent?

Preliminaries



Morphisms: objects of type of i (raw domain D)

Partial functions:

domain	dom	of type $i \to i$
codomain	cod	of type $i \to i$
composition	$\cdot$	of type $i \to i \to i$ (resp. $i \times i \to i$)

$\cong$ denotes Kleene equality: $x \cong y \equiv (Ex \vee Ey) \to x = y$

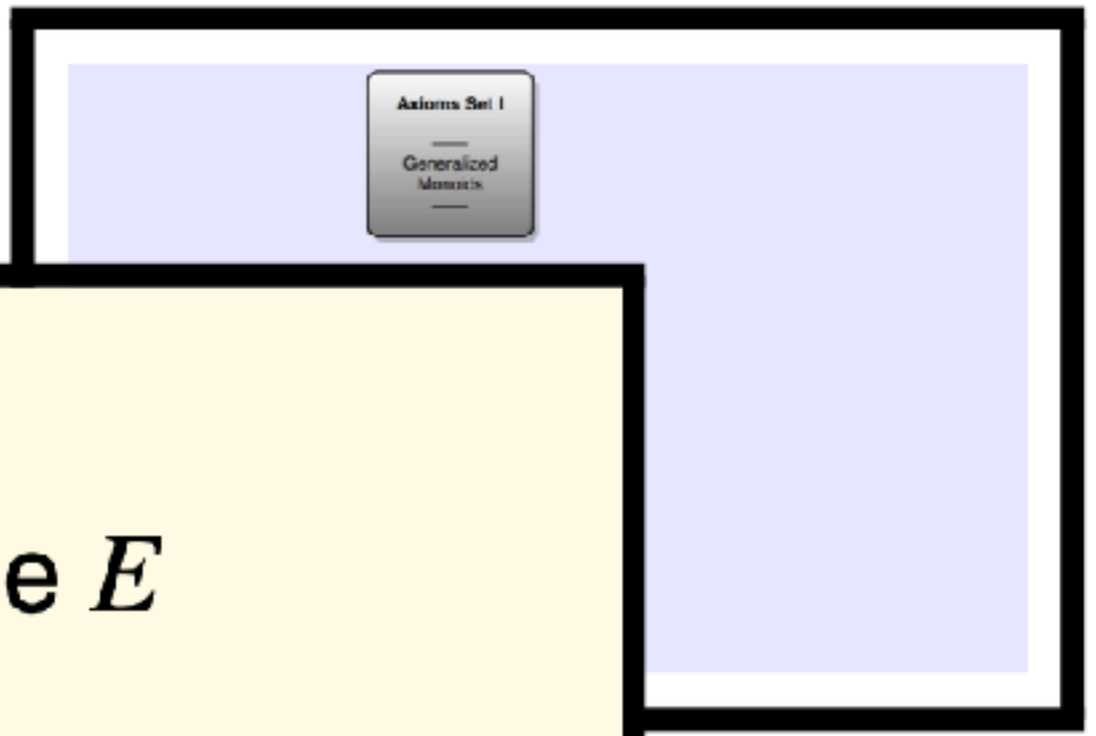
(where $=$ is identity on all objects of type i, existing or non-existing)

$\cong$ is an equivalence relation: **Sledgehammer**.

$\simeq$ denotes existing identity: $x \simeq y \equiv Ex \wedge Ey \wedge x = y$

$\simeq$ is symmetric and transitive, but lacks reflexivity: **Sledgehammer**, **Nitpick**.

Preliminaries



- $\simeq$ equivalence relation on E, empty relation outside E
- $1/0 \not\simeq 1/0 \quad 1/0 \not\simeq 2/0 \quad \ldots$
- $Ix.pkoFrance(x) \not\simeq Ix.pkoFrance(x)$
  $Ix.pkoFrance(x) \not\simeq Ix.pkoPoland(x)$

$\cong$ denotes Kleene equality: $x \cong y \equiv (Ex \vee Ey) \rightarrow x = y$

(where $=$ is identity on all objects of type i, existing or non-existing)

$\cong$ is an equivalence relation: **Sledgehammer**.

$\simeq$ denotes existing identity: $x \simeq y \equiv Ex \wedge Ey \wedge x = y$

$\simeq$ is symmetric and transitive, but lacks reflexivity: **Sledgehammer**, **Nitpick**.

Category Theory in Free Logic (in HOL)

```
section ‹ Axioms Set V ›

locale Axioms_Set_V =
assumes
  S1: "E(dom x) ⟶ E x" and
  S2: "E(cod y) ⟶ E y" and
  S3: "E(x⋅y) ↔ dom x ≃ cod y" and
  S4: "x⋅(y⋅z) ≅ (x⋅y)⋅z" and
  S5: "x⋅(dom x) ≅ x" and
  S6: "(cod y)⋅y ≅ y"
begin  (*The obligatory consistency checks*)
  lemma True
     nitpick [satisfy, user_axioms, expect=genuine] oops (*model found*)
  lemma assumes "∃x. ¬(E x)" shows True
    nitpick [satisfy, user_axioms, expect=genuine] oops (*model found*)
  lemma assumes "(∃x. ¬(E x)) ∧ (∃x. (E x))" shows True
    nitpick [satisfy, user_axioms,  expect=genuine] oops (*model found*)
end

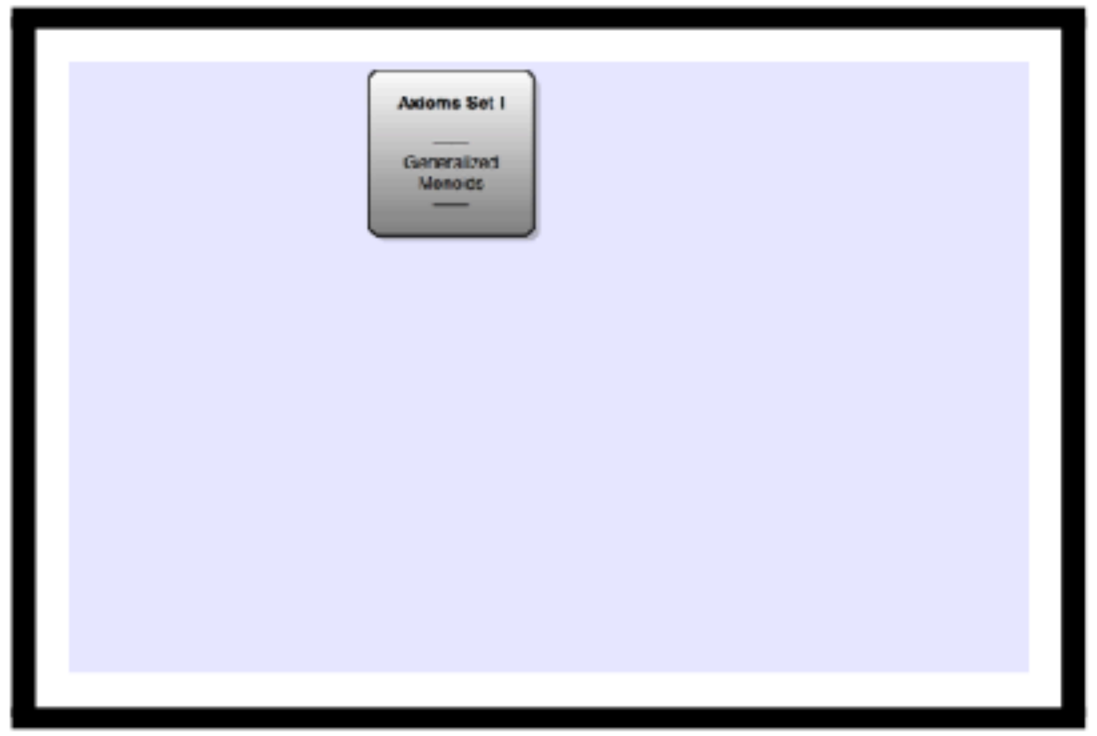
context Axioms_Set_V   (*Axioms Set IV is implied by Axioms Set V*)
begin
  lemma SivFromV: "(E(x⋅y) ⟶ (E x ∧ E y)) ∧ (E(dom x ) ⟶ E x) ∧ (E(cod y) ⟶ E y)"
    using S1 S2 S3 by blast
  lemma EivFromV: "E(x⋅y) ↔ (dom x ≅ cod y ∧ E(cod y))"  using S3 by metis
  lemma AivFromV: "x⋅(y⋅z) ≅ (x⋅y)⋅z"  using S4 by blast
  lemma CivFromV: "(cod y)⋅y ≅ y"  using S6 by blast
  lemma DivFromV: "x⋅(dom x) ≅ x"  using S5 by blast
end

context Axioms_Set_IV  (*Axioms Set V is implied by Axioms Set IV*)
begin
  lemma S1FromIV:  "E(dom x) ⟶ E x"  using Siv by blast
  lemma S2FromIV:  "E(cod y) ⟶ E y"  using Siv by blast
  lemma S3FromIV:  "E(x⋅y) ↔ dom x ≃ cod y"  using Eiv by metis
  lemma S4FromIV:  "x⋅(y⋅z) ≅ (x⋅y)⋅z"  using Aiv by blast
  lemma S5FromIV:  "x⋅(dom x) ≅ x"  using Div by blast
  lemma S6FromIV:  "(cod y)⋅y ≅ y"  using Civ by blast
```

Table 1 Stepwise evolution of Scott's [33] axiom system for category theory from partial monoids

Axioms Set I	
S_i	$E(x\cdot y) \longrightarrow (Ex \wedge Ey)$
E_i	$E(x\cdot y) \longleftarrow (Ex \wedge Ey \wedge (\exists z. z\cdot z \cong z \wedge x\cdot z \cong x \wedge z\cdot y \cong y))$
A_i	$x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$
C_i	$\forall y. \exists i. Ii \wedge i\cdot y \cong y$
D_i	$\forall x. \exists j. Ij \wedge x\cdot j \cong x$
Axioms Set II	
S_{ii}	$E(x\cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\, x) \longrightarrow Ex) \wedge (E(cod\, y) \longrightarrow Ey)$
E_{ii}	$E(x\cdot y) \longleftarrow (Ex \wedge Ey \wedge (\exists z. z\cdot z \cong z \wedge x\cdot z \cong x \wedge z\cdot y \cong y))$
A_{ii}	$x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$
C_{ii}	$Ey \longrightarrow (I(cod\, y) \wedge (cod\, y)\cdot y \cong y)$
D_{ii}	$Ex \longrightarrow (I(dom\, x) \wedge x\cdot(dom\, x) \cong x)$
Axioms Set III	
S_{iii}	$E(x\cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\, x) \longrightarrow Ex) \wedge (E(cod\, y) \longrightarrow Ey)$
E_{iii}	$E(x\cdot y) \longleftarrow (dom\, x \cong cod\, y \wedge E(cod\, y)))$
A_{iii}	$x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$
C_{iii}	$Ey \longrightarrow (I(cod\, y) \wedge (cod\, y)\cdot y \cong y)$
D_{iii}	$Ex \longrightarrow (I(dom\, x) \wedge x\cdot(dom\, x) \cong x)$
Axioms Set IV	
S_{iv}	$E(x\cdot y) \longrightarrow (Ex \wedge Ey) \wedge (E(dom\, x) \longrightarrow Ex) \wedge (E(cod\, y) \longrightarrow Ey)$
E_{iv}	$E(x\cdot y) \longleftrightarrow (dom\, x \cong cod\, y \wedge E(cod\, y)))$
A_{iv}	$x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$
C_{iv}	$(cod\, y)\cdot y \cong y$
D_{iv}	$x\cdot(dom\, x) \cong x$
Axioms Set V [33]	
$S1$	$E(dom\, x) \longrightarrow Ex$
$S2$	$E(cod\, y) \longrightarrow Ey$
$S3$	$E(x\cdot y) \longleftrightarrow dom\, x \simeq cod\, y$
$S4$	$x\cdot(y\cdot z) \cong (x\cdot y)\cdot z$
$S5$	$(cod\, y)\cdot y \cong y$
$S6$	$x\cdot(dom\, x) \cong x$

From Monoids to Categories



We employ a partial, strict binary composition operation $\cdot$
Left and right identity elements are addressed in C_i, D_i, .

Categories: Axioms Set I

S_i	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey)$
E_i	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_i	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_i	Codomain	$\forall y. \exists i. ID(i) \wedge i \cdot y \cong y$
D_i	Domain	$\forall x. \exists j. ID(j) \wedge x \cdot j \cong x$

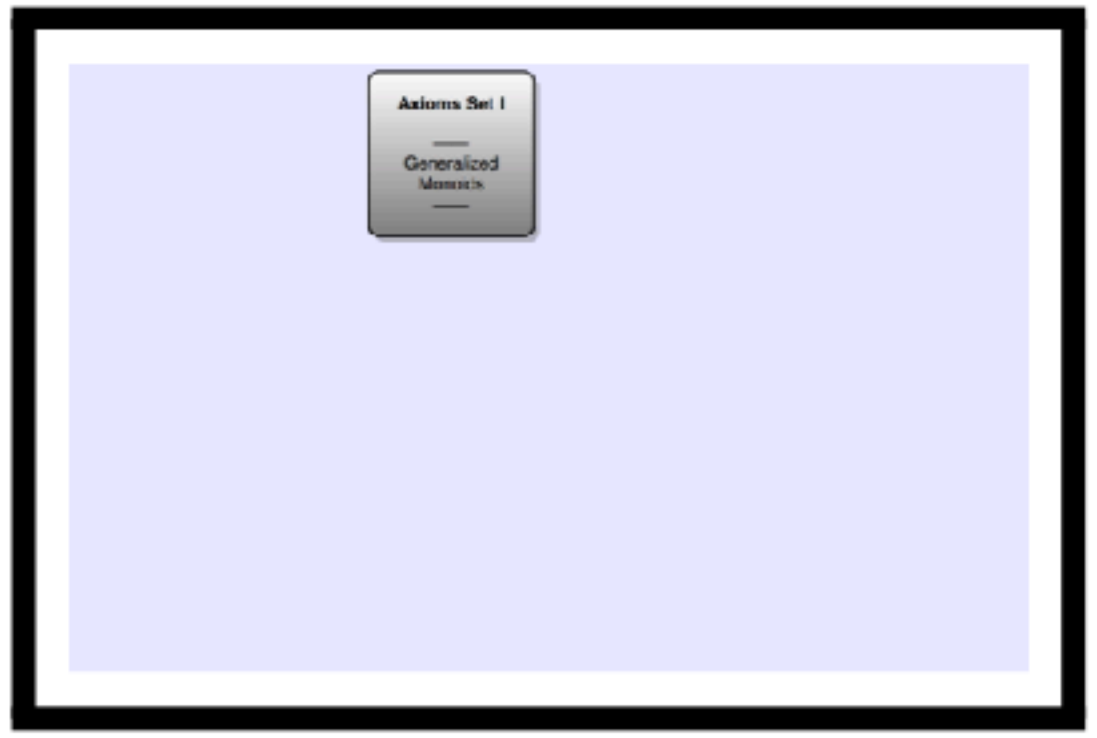
where I is an identity morphism predicate:

$$ID(i) \equiv (\forall x.\ E(i \cdot x) \rightarrow i \cdot x \cong x) \wedge (\forall x.\ E(x \cdot i) \rightarrow x \cdot i \cong x)$$

Monoid

Closure:	$\forall a, b \in S.\ a \circ b \in S$
Associativity:	$\forall a, b, c \in S.\ a \circ (b \circ c) = (a \circ b) \circ c$
Identity:	$\exists id_S \in S.\ \forall a \in S.\ id_S \circ a = a = a \circ id_S$

From Monoids to Categories



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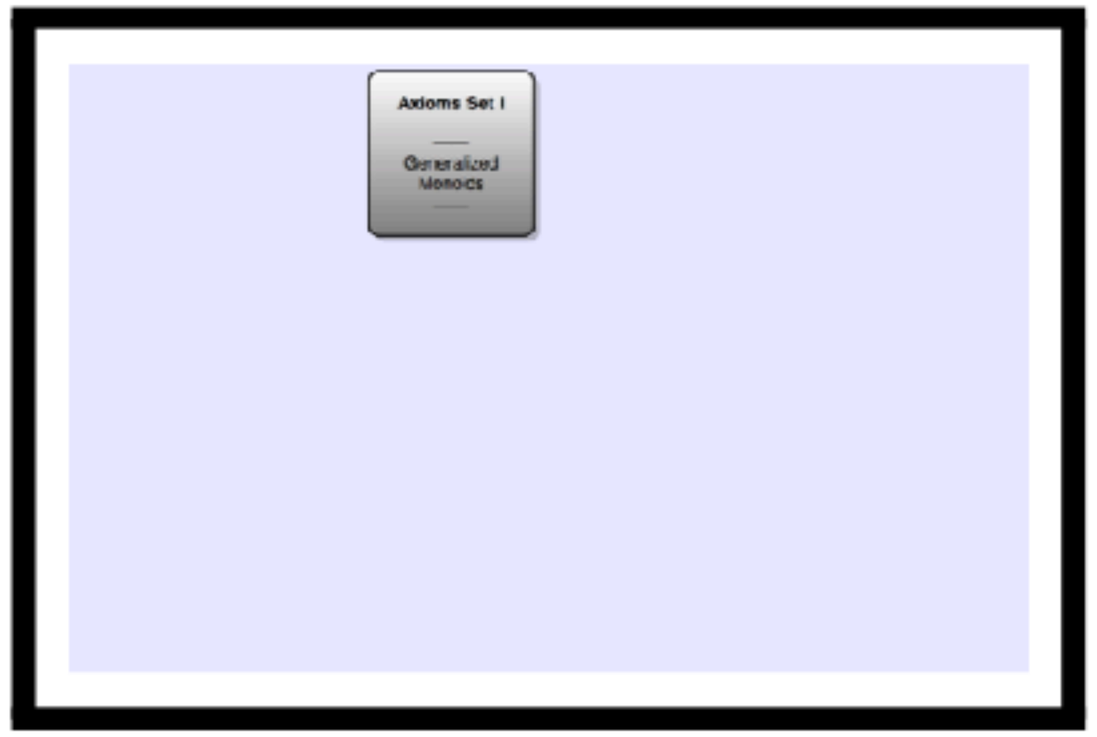
Experiments with Isabelle/HOL

- The i in axiom C is unique: **Sledgehammer**.
- The j in axiom D is unique: **Sledgehammer**.
- However, the i and j need not be equal: **Nitpick**

From Monoids to Categories



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A_i	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_i	Codomain	$\forall y.\exists i.ID(i) \wedge i \cdot y \cong y$
D_i	Domain	$\forall x.\exists j.ID(j) \wedge x \cdot j \cong x$

where I is an identity morphism predicate:

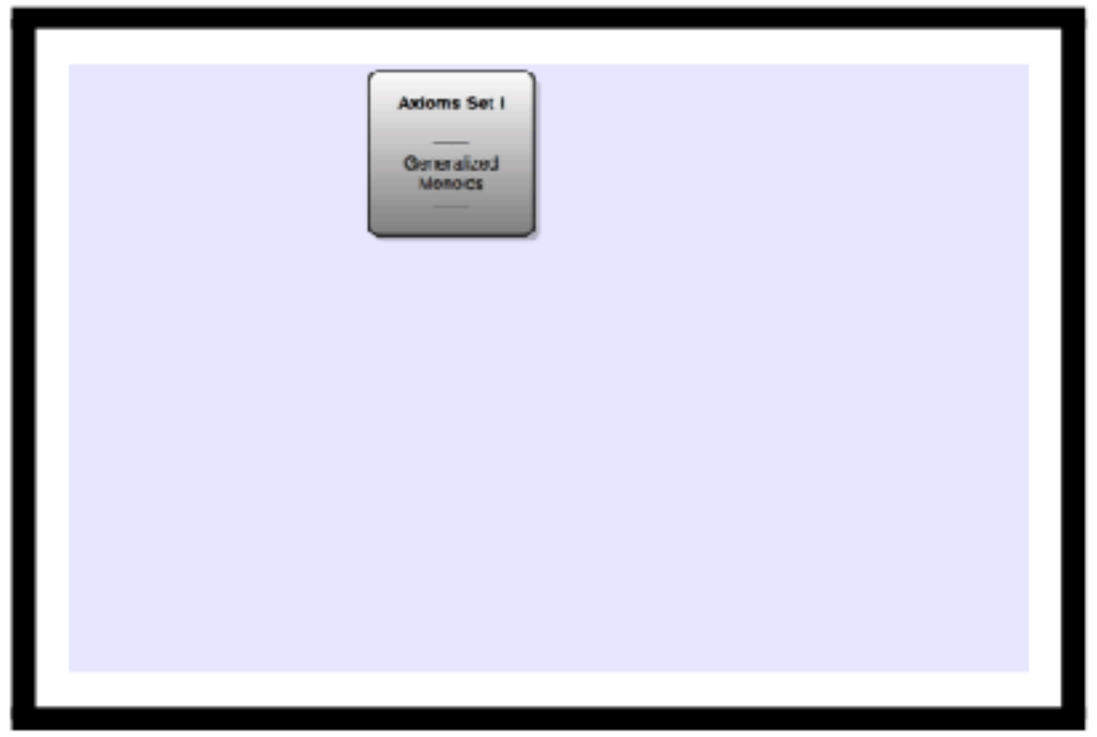
$$ID(i) \equiv (\forall x.\ E(i \cdot x) \rightarrow i \cdot x \cong x) \wedge (\forall x.\ E(x \cdot i) \rightarrow x \cdot i \cong x)$$

Experiments with Isabelle/HOL

- The left-to-right direction of E is implied: **Sledgehammer**.

$$E(x \cdot y) \rightarrow (Ex \wedge Ey \wedge (\exists z.z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$$

From Monoids to Categories



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A_i	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
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where I is an identity morphism predicate:

$$ID(i) \equiv (\forall x.\ E(i \cdot x) \rightarrow i \cdot x \cong x) \wedge (\forall x.\ E(x \cdot i) \rightarrow x \cdot i \cong x)$$

Experiments with Isabelle/HOL

- Model finder **Nitpick** confirms that this axiom set is consistent.
- Even if we assume there are non-existing objects ($\exists x. \neg(Ex)$) we get consistency.

Interaction: Dana – Christoph – Isabelle/HOL

Dana Scott <dana.scott@cs.cmu.edu> 8/6/16
to me

> On Aug 5, 2016, at 11:00 PM, Christoph Benzmueller
>
> When we take IDD(i) as
>   (all x)[E(i.x) ==> i.x == x] &
>   (all x)[E(x.i) ==> x.i == x]
> and replace ID(i) in our SACDE-axioms by IDD(i) the
> ID(I) and IDD(i) are equivalent. See attachment New_
>
> So IDD(i) seem suited as a notion of identity morphis

Ha! I am surprised, because I did not see how to prove

(all i)[IDD(i) ==> i.i == i]

I have to think about this. I hate it when computers are smarter than I am!

I guess C and D have to be used.

Chri
to D
Hi D
C.

Existing: 1, 2   Undefined: 3

	dom	cod
1	1	1
2	2	2
3	3	3

·	1	2	3
1	1	3	3
2	3	2	3
3	3	3	3

Interaction: Dana – Christoph – Isabelle/HOL

Christoph Benzmueller <c.benzmueller@gmail.com> 7/23/16
to Dana

Dana,

here are the results of the experiments; doesn't look too good.

On Fri, Jul 22, 2016 at 11:43 PM, Dana Scott <dana.scott@cs.cmu.edu> wrote:

> On Jul 21, 2016, at 9:32 AM, Christoph Benzmueller <c.benzmueller@gmail.com> wrote:
>
> The F-axioms are all provable from the old S-axioms.
> But D2, D3 and E3 are not.

I think I see the trouble with those D axioms. But E3 is very odd.

E3: E(x.y) ==> (exist i)[Id(i) & x.(i.y) == x.y]

You see, by the S-axioms, if you assume E(x.y), then E(x) & E(y) & E(cod(x)) follows. So the "i" in the conclusion of E3 ought to be "cod(x)".

Please check, therefore, whether this is provable from the S-axioms:

(all x) Id(cod(x))

Apparently it isn't. See file Scott_new_axioms_4.png; the countermodel is presented in the lower window; he have:

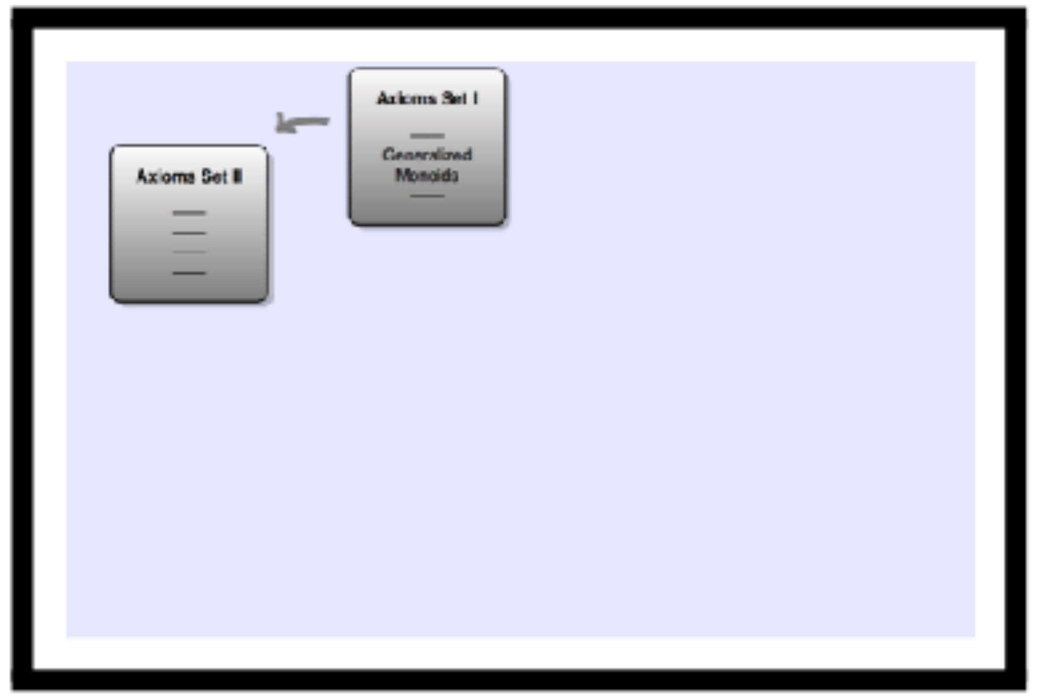
dom(i1)=i1, dom(i2)=i2, dom(i3)=i3
cod(i1)=i1, cod(i2)=i2, cod(i3)=i3
i1.i1=i1, i1.i2=i3, i1.i3=i3
i2.i1=i3, i2.i2=i2, i2.i3=i3
i3.i1=i3, i3.i2=i3, i3.i3=i3
E(i1),E(i2), ~E(i3)

Countermodel by Nitpick converted by me into a readable form

I have briefly checked it; it seems to validate each S-axiom.

If this is OK, then E3 should have been provable.

From Monoids to Categories



Axioms Set II is developed from Axioms Set I by Skolemization of i and j in axioms C and D. We can argue semantically that every model of Axioms Set I has such functions. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions dom and cod.

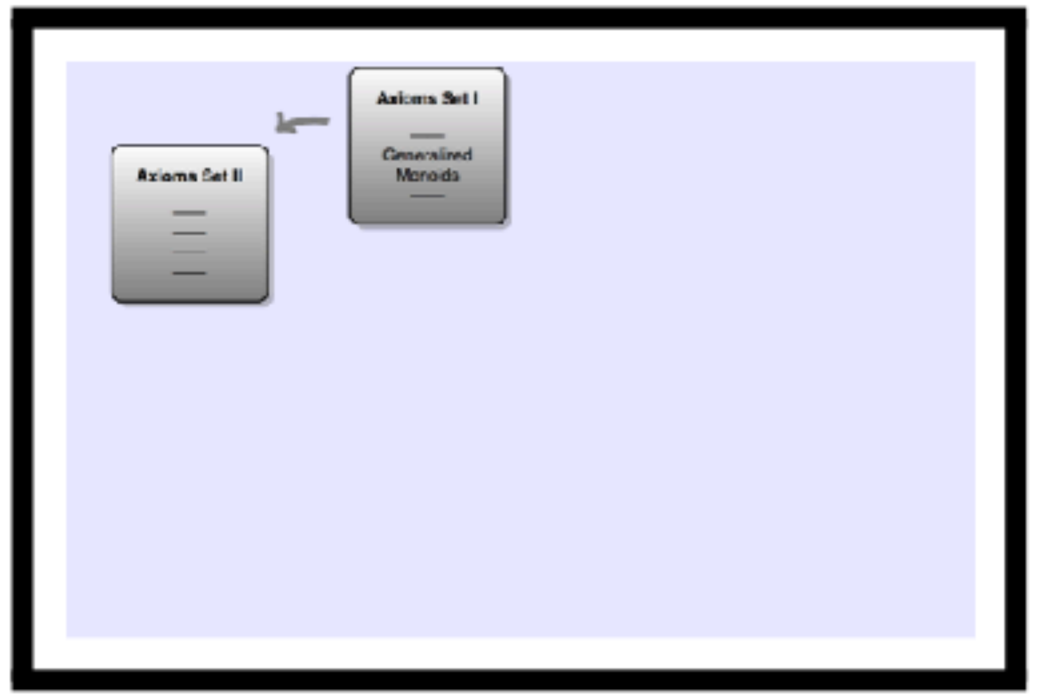
Categories: Axioms Set II

S_{ii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{ii}	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_{ii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{ii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{ii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Categories: Axioms Set I

S_i	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey)$
E_i	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_i	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_i	Codomain	$\forall y. \exists i. ID(i) \wedge i \cdot y \cong y$
D_i	Domain	$\forall x. \exists j. ID(j) \wedge x \cdot j \cong x$

From Monoids to Categories



Axioms Set II is developed from Axioms Set I by Skolemization of i and j in axioms C and D. We can argue semantically that every model of Axioms Set I has such functions. The strictness axiom S is extended, so that strictness is now also postulated for the new Skolem functions dom and cod.

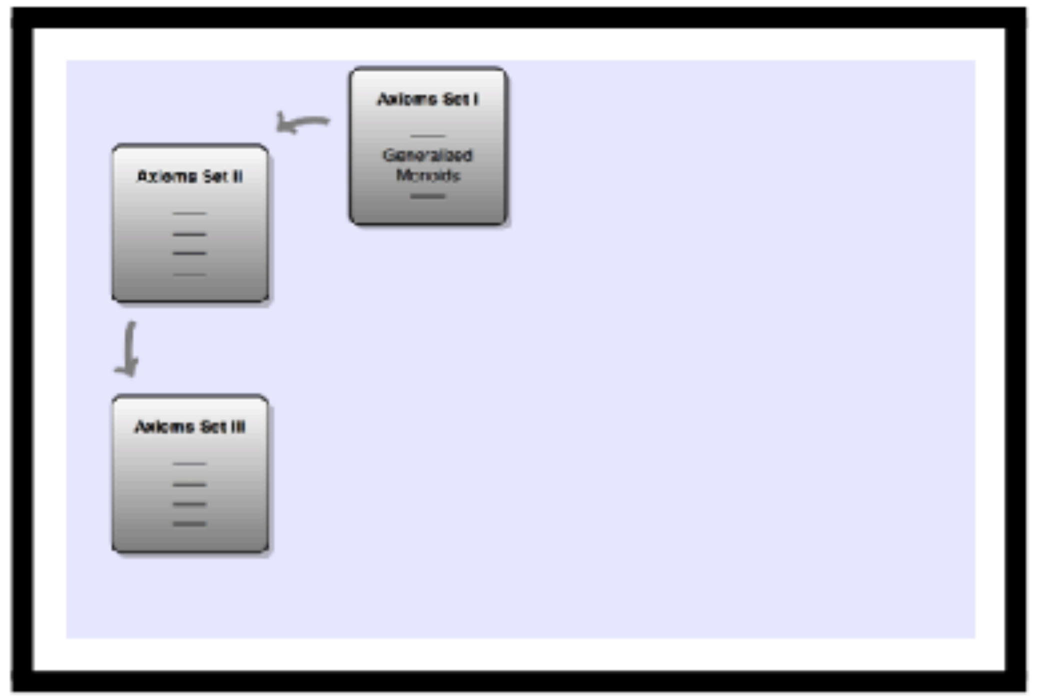
Categories: Axioms Set II

S_{ii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{ii}	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_{ii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{ii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{ii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set II implies Axioms Set I: easily proved by **Sledgehammer**.
- Axioms Set I also implies Axioms Set II (by semantical means on the meta-level)

From Monoids to Categories



In Axioms Set III the existence axiom E is simplified by taking advantage of the two new Skolem functions dom and cod.

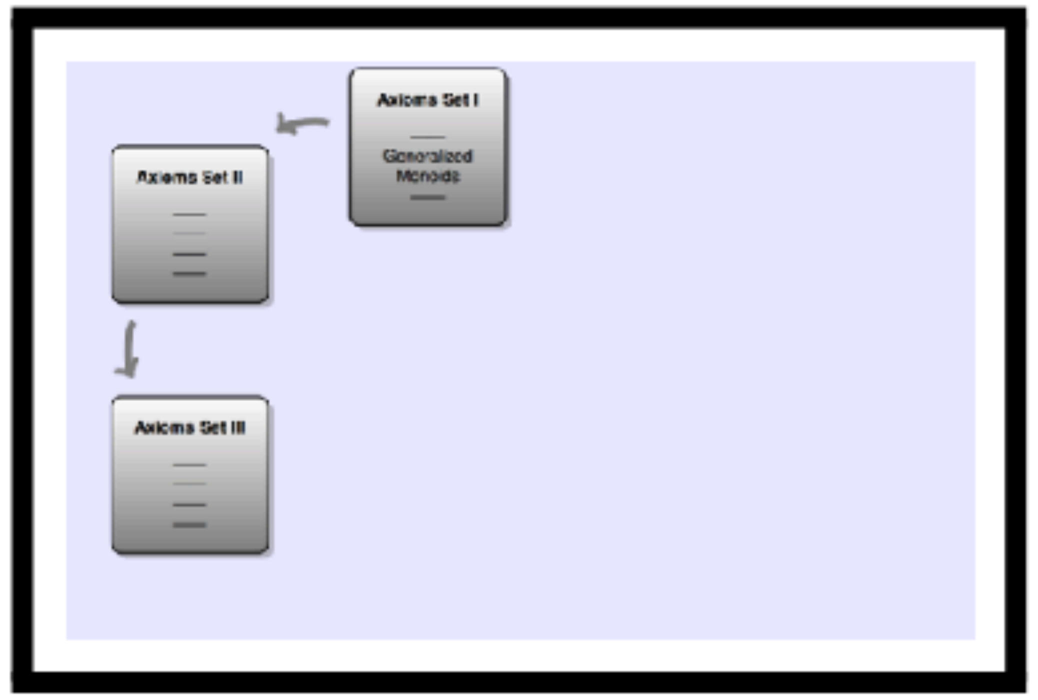
Categories: Axioms Set III

S_{iii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iii}	Existence	$E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{iii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

Categories: Axioms Set II

S_{ii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{ii}	Existence	$E(x \cdot y) \leftarrow (Ex \wedge Ey \wedge (\exists z. z \cdot z \cong z \wedge x \cdot z \cong x \wedge z \cdot y \cong y))$
A_{ii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{ii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{ii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

From Monoids to Categories



In Axioms Set III the existence axiom E is simplified by taking advantage of the two new Skolem functions dom and cod.

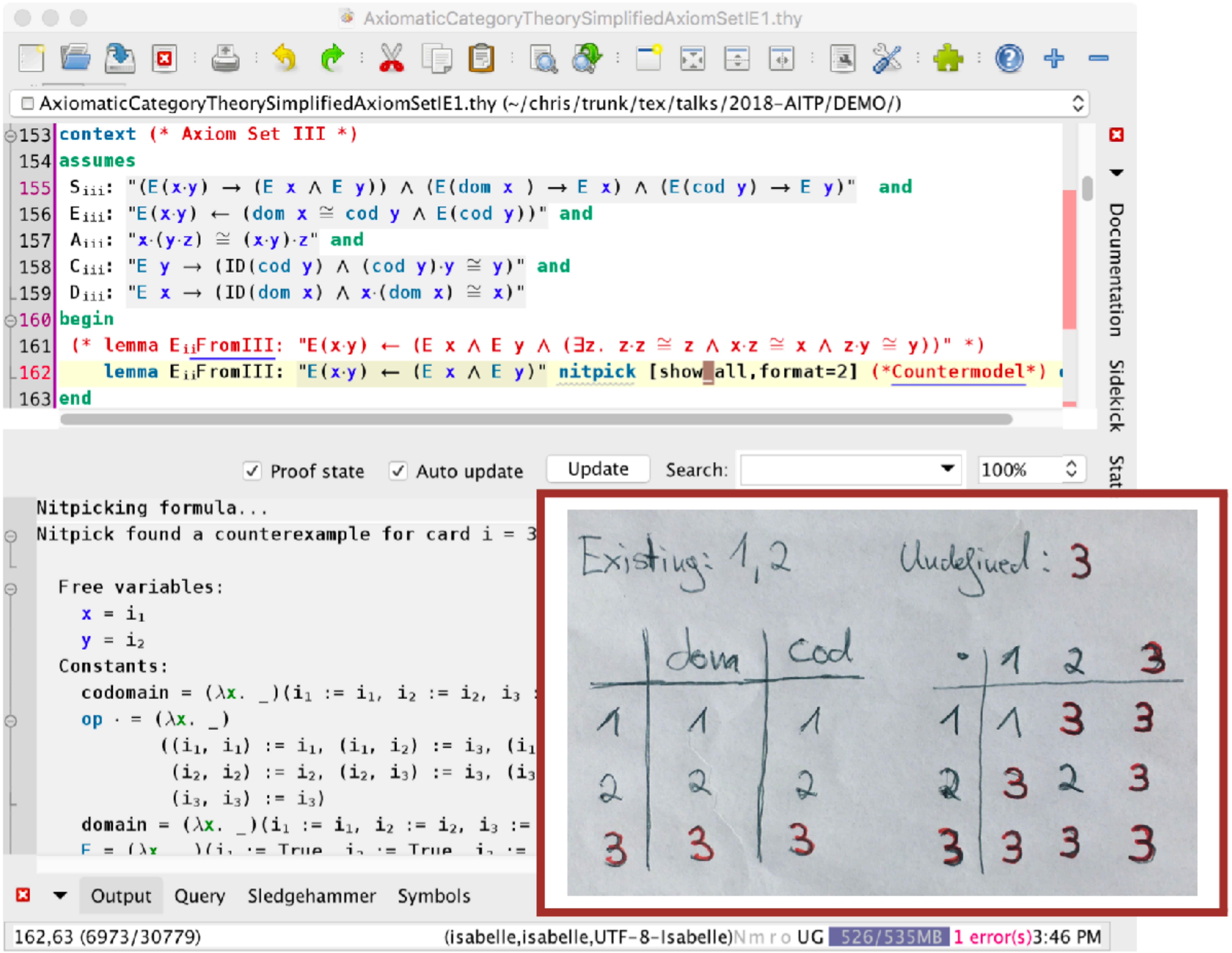
Categories: Axioms Set III

S_{iii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iii}	Existence	$E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{iii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

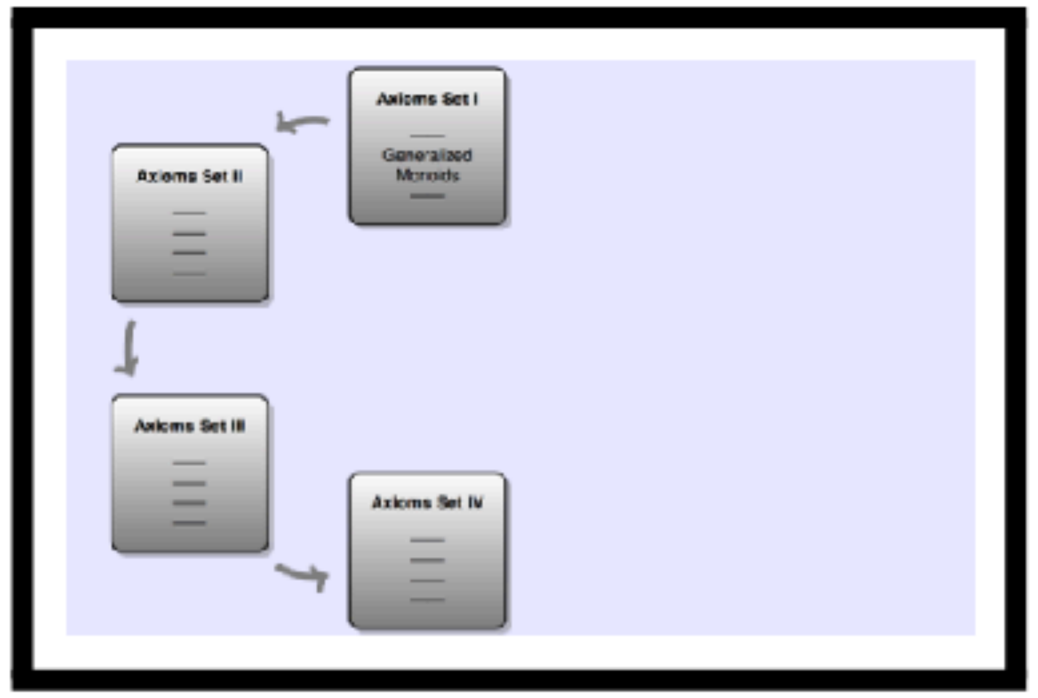
Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- The left-to-right direction of existence axiom E is implied: **Sledgehammer**.
- Axioms Set III implies Axioms Set II: **Sledgehammer**.
- Axioms Set II implies Axioms Set III: **Sledgehammer**.

Interesting Model (idempotents, but no left- & right-identities)



From Monoids to Categories



Axioms Set IV simplifies the axioms C and D. However, as it turned out, these simplifications also require the existence axiom E to be strengthened into an equivalence.

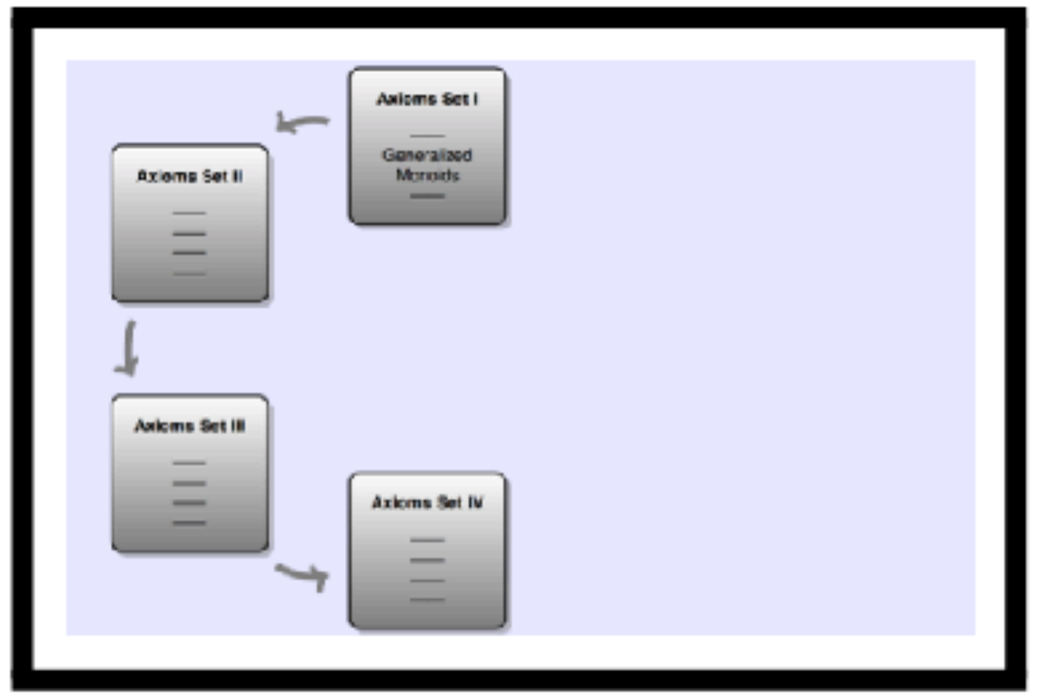
Categories: Axioms Set IV

S_{iv}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iv}	Existence	$E(x \cdot y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iv}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iv}	Codomain	$(cod\ y) \cdot y \cong y$
D_{iv}	Domain	$x \cdot (dom\ x) \cong x$

Categories: Axioms Set III

S_{iii}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iii}	Existence	$E(x \cdot y) \leftarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iii}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iii}	Codomain	$Ey \rightarrow (ID(cod\ y) \wedge (cod\ y) \cdot y \cong y)$
D_{iii}	Domain	$Ex \rightarrow (ID(dom\ x) \wedge x \cdot (dom\ x) \cong x)$

From Monoids to Categories



Axioms Set IV simplifies the axioms C and D. However, as it turned out, these simplifications also require the existence axiom E to be strengthened into an equivalence.

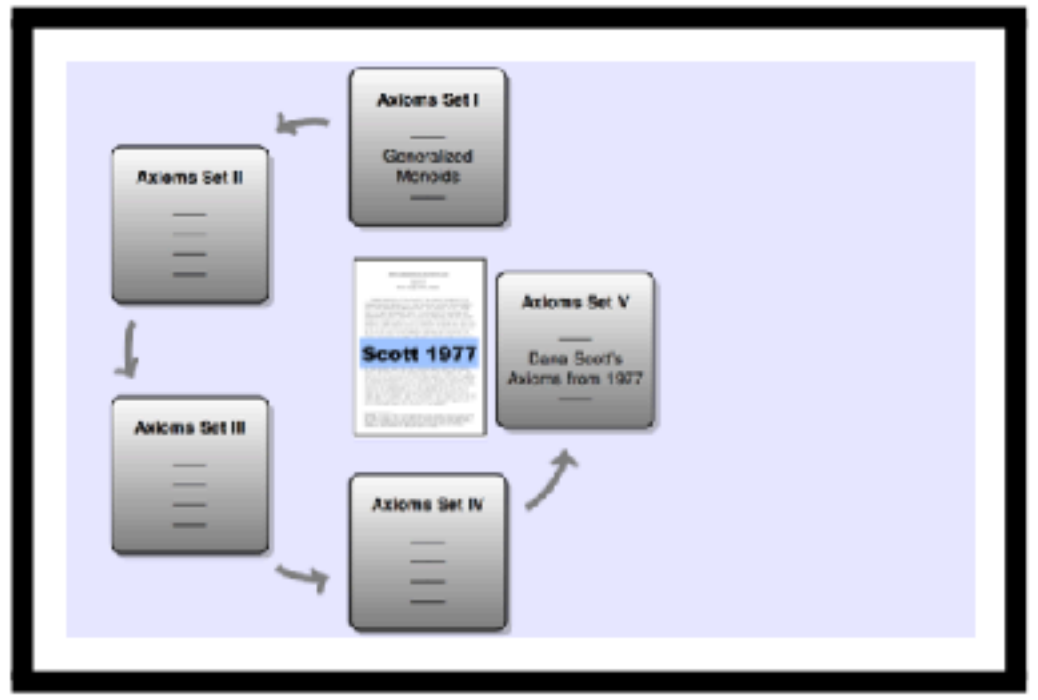
Categories: Axioms Set IV

S_{iv}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iv}	Existence	$E(x \cdot y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iv}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iv}	Codomain	$(cod\ y) \cdot y \cong y$
D_{iv}	Domain	$x \cdot (dom\ x) \cong x$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set IV implies Axioms Set III: **Sledgehammer**.
- Axioms Set III implies Axioms Set IV: **Sledgehammer**.

From Monoids to Categories



Axioms Set V simplifies axiom E (and S).
Now, strictness of $\cdot$ is implied.

Categories: Axioms Set V (Scott, 1977)

$S1$	Strictness	$E(dom\ x) \rightarrow Ex$
$S2$	Strictness	$E(cod\ y) \rightarrow Ey$
$S3$	Existence	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
$S4$	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
$S5$	Codomain	$(cod\ y) \cdot y \cong y$
$S6$	Domain	$x \cdot (dom\ x) \cong x$

Categories: Axioms Set IV

S_{iv}	Strictness	$E(x \cdot y) \rightarrow (Ex \wedge Ey) \wedge (E(dom\ x) \rightarrow Ex) \wedge (E(cod\ y) \rightarrow Ey)$
E_{iv}	Existence	$E(x \cdot y) \leftrightarrow (dom\ x \cong cod\ y \wedge E(cod\ y))$
A_{iv}	Associativity	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$
C_{iv}	Codomain	$(cod\ y) \cdot y \cong y$
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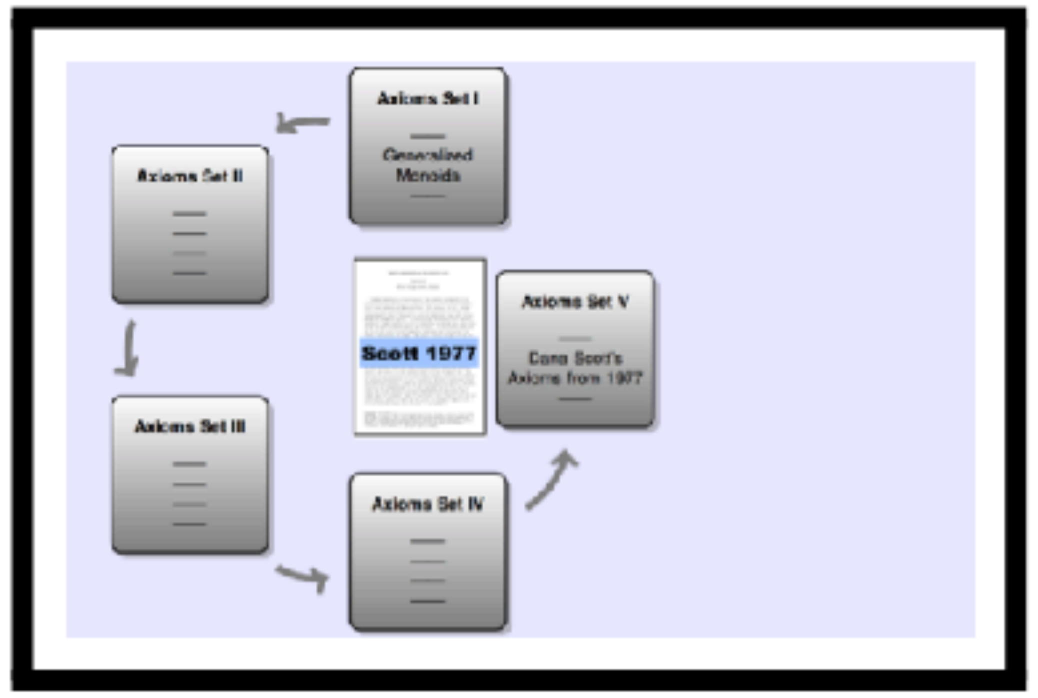
From Monoids to Categories



Axioms Set V simplifies axiom E (and S).
Now, strictness of $\cdot$ is implied.

Categories: Axioms Set V (Scott, 1977)

$S1$	Strictness	$E(dom\ x) \rightarrow Ex$
$S2$	Strictness	$E(cod\ y) \rightarrow Ey$
$S3$	Existence	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
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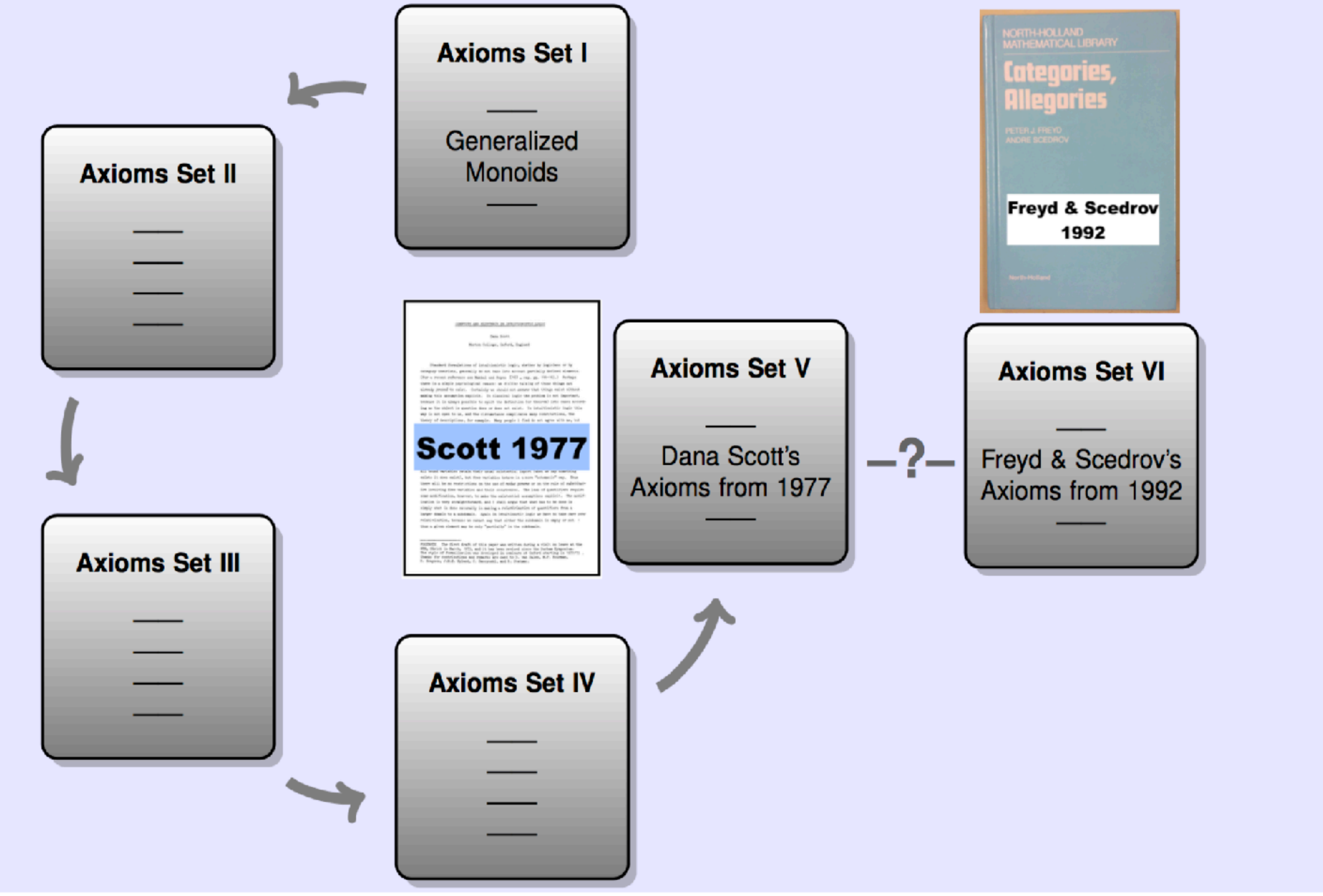
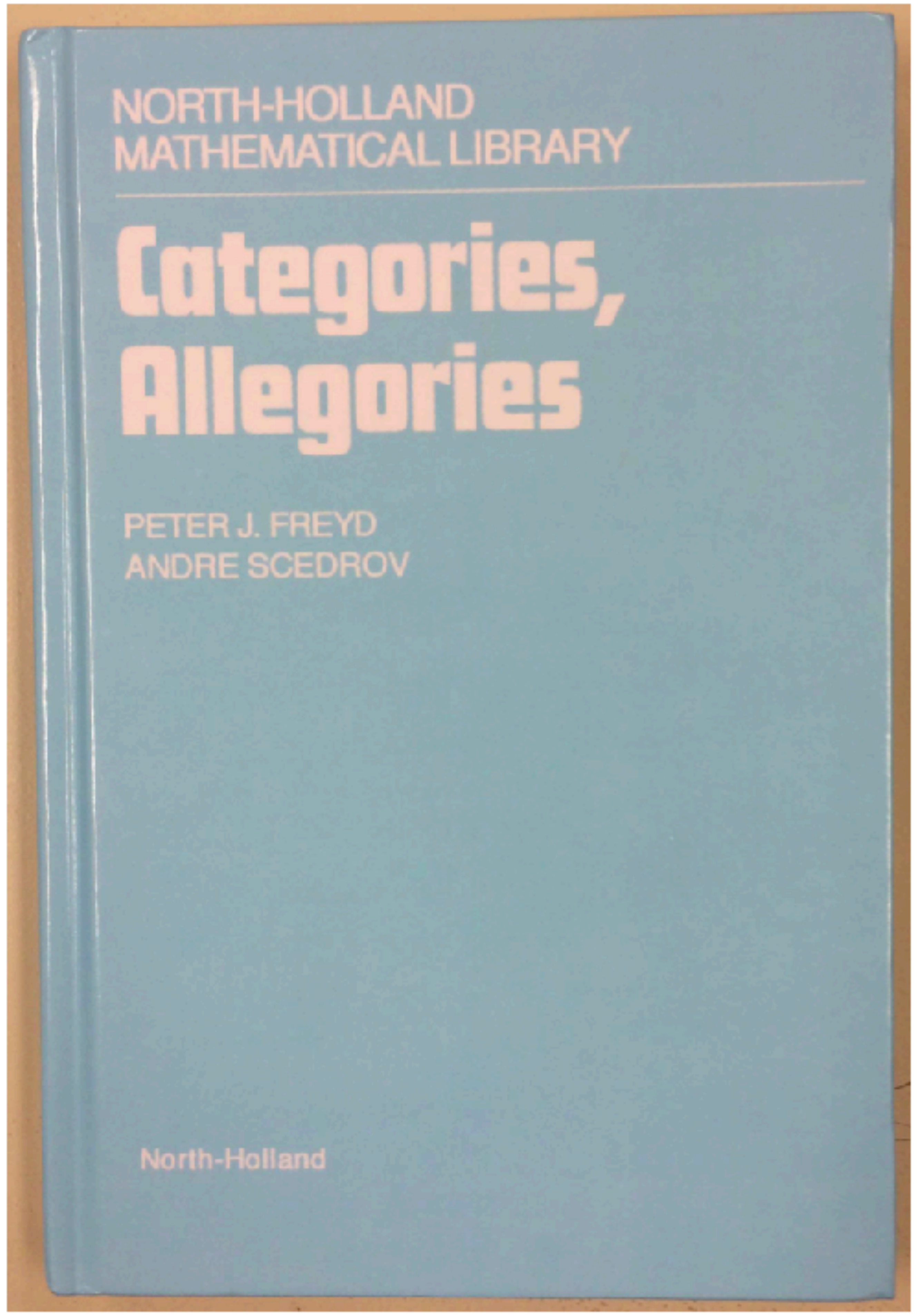
Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set V implies Axioms Set IV: **Sledgehammer**.
- Axioms Set IV implies Axioms Set V: **Sledgehammer**.

Cats & Alligators



Cats & Alligators



1.1. BASIC DEFINITIONS

The theory of CATEGORIES is given by two unary operations and a binary partial operation. In most contexts lower-case variables are used for the 'individuals' which are called *morphisms* or *maps*. The values of the operations are denoted and pronounced as:

$\Box x$ *the source of* x ,

$x\Box$ *the target of* x ,

xy *the composition of* x *and* y .

The axioms:

A1 xy *is defined* *iff* $x\Box = \Box y$,

A2a $(\Box x)\Box = \Box x$ *and* $\Box(x\Box) = x\Box$, A2b

A3a $(\Box x)x = x$ *and* $x(x\Box) = x$, A3b

A4a $\Box(xy) = \Box(x(\Box y))$ *and* $(xy)\Box = ((x\Box)y)\Box$, A4b

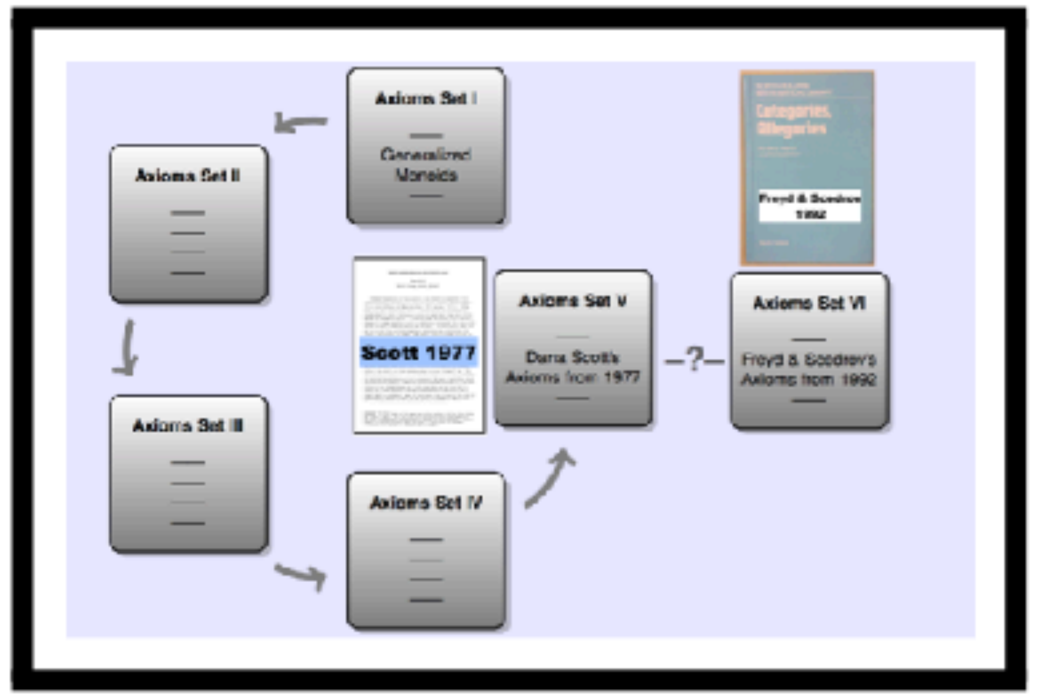
A5 $x(yz) = (xy)z$.

1.11. The ordinary equality sign = will be used only in the symmetric sense, to wit: if either side is defined then so is the other and they are equal. A theory, such as this, built on an ordered list of partial operations, the domain of definition of each given by equations in the previous, and with all other axioms equational, is called an ESSENTIALLY ALGEBRAIC THEORY.

1.12. We shall use a venturi-tube $\succcurlyeq$ for *directed equality* which means: if the left side is defined then so is the right and they are equal. The axiom that $\Box(xy) = \Box(x(\Box y))$ is equivalent, in the presence of the earlier axioms, with $\Box(xy) \succcurlyeq \Box x$ as can be seen below.

1.13. $\Box(\Box x) = \Box x$ because $\Box(\Box x) = \Box((\Box x)\Box) = (\Box x)\Box = \Box x$. Similarly $(x\Box)\Box = x\Box$.

Cats & Alligators

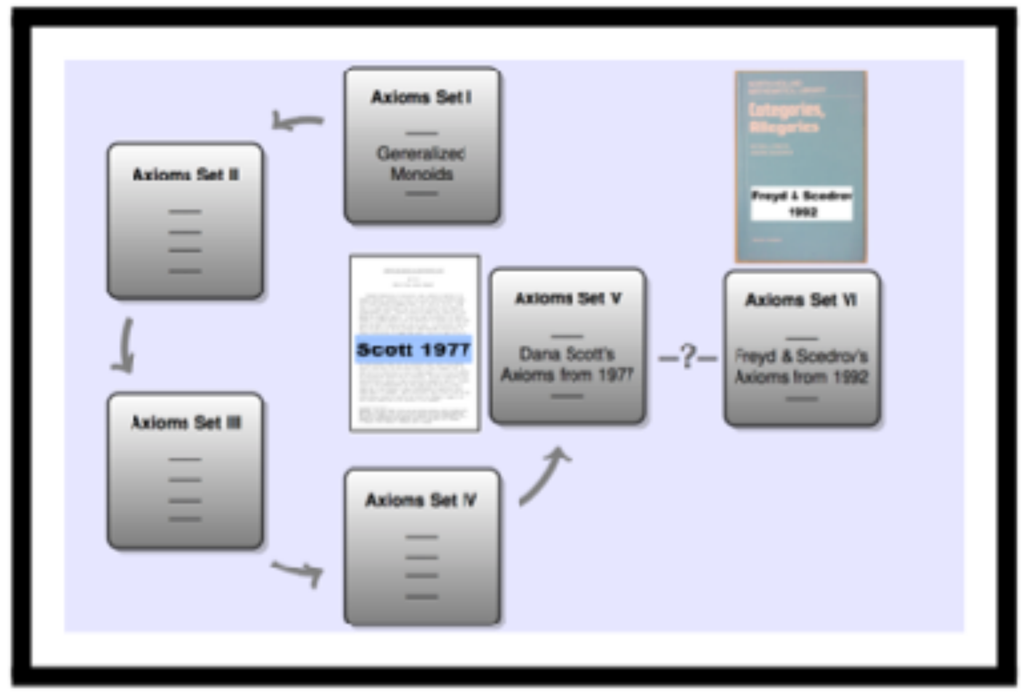


Categories: Original axiom set by Freyd and Scedrov (modulo notation)

A1	$E(x \cdot y) \leftrightarrow dom\ x \cong cod\ y$
A2a	$cod(dom\ x) \cong dom\ x$
A2b	$dom(cod\ y) \cong cod\ y$
A3a	$x \cdot (dom\ x) \cong x$
A3b	$(cod\ y) \cdot y \cong y$
A4a	$dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

- Consistency? — Nitpick finds a model.
- Consistency when assuming $\exists x.\neg Ex$ — Nitpick does not find a model.
- lemma $(\exists x.\neg Ex) \rightarrow False$: **Sledgehammer**. (Problematic axioms: $A1, A2a, A3a$)



Categories: Axioms Set VI (Freyd and Scedrov, when corrected)

A1	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
A2a	$cod(dom\ x) \cong dom\ x$
A2b	$dom(cod\ y) \cong cod\ y$
A3a	$x \cdot (dom\ x) \cong x$
A3b	$(cod\ y) \cdot y \cong y$
A4a	$dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

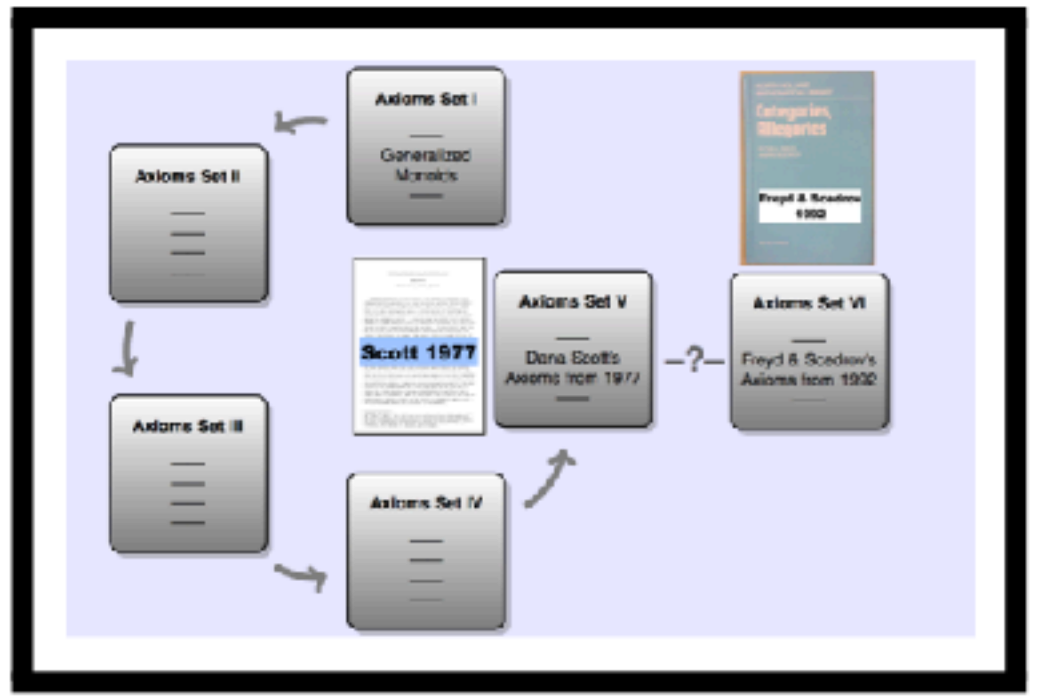
Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.
- Axioms Set VI implies Axioms Set V: **Sledgehammer**.
- Axioms Set V implies Axioms Set VI: **Sledgehammer**.
- Redundancies:

— The $A4$-axioms are implied by the others: **Sledgehammer**.

— The $A2$-axioms are implied by the others: **Sledgehammer**.

Cats & Alligators



Maybe Freyd and Scedrov do not assume a free logic. In algebraic theories free variables often range over existing objects only. However, we can formalise this as well:

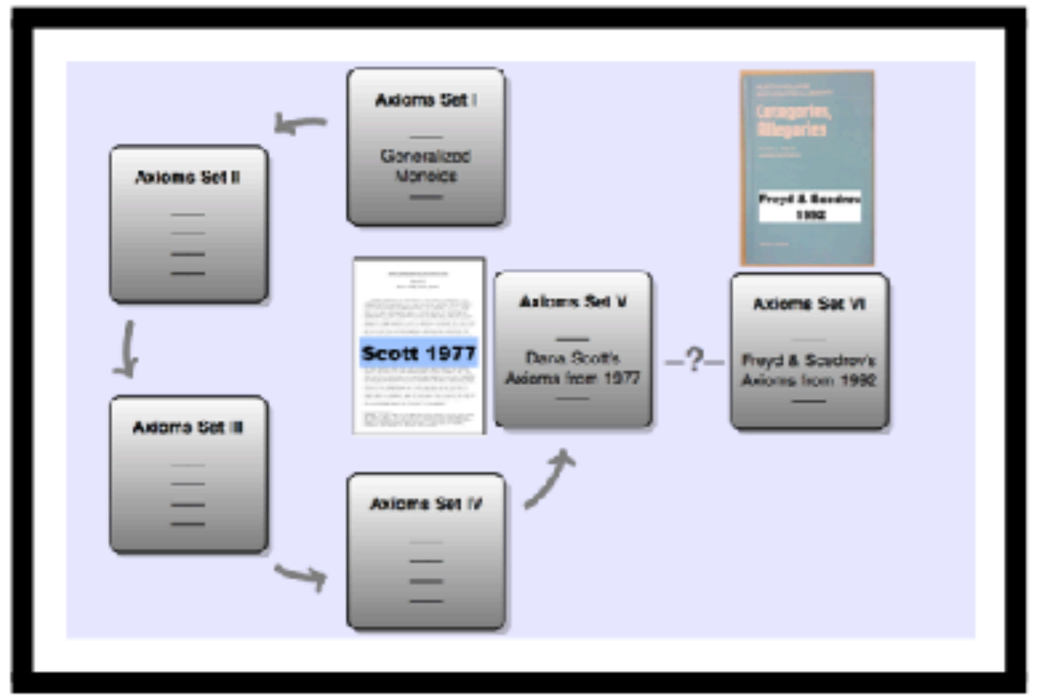
Categories: "Algebraic reading" of axiom set by Freyd and Scedrov.

A1	$\forall xy.\ E(x \cdot y) \leftrightarrow dom\ x \cong cod\ y$
A2a	$\forall x.\ cod(dom\ x) \cong dom\ x$
A2b	$\forall y.\ dom(cod\ y) \cong cod\ y$
A3a	$\forall x.\ x \cdot (dom\ x) \cong x$
A3b	$\forall y.\ (cod\ y) \cdot y \cong y$
A4a	$\forall xy.\ dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$\forall xy.\ cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$\forall xyz.\ x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

- Consistency holds (also when $\exists x.\neg(Ex)$): confirmed by **Nitpick**.
- However, none of V-axioms are implied: **Nitpick**.
- For equivalence to V-axioms: add strictness of dom, cod, $\cdot$, **Sledgehammer**.

Cats & Alligators



Maybe Freyd and Scedrov do not assume a free logic.
In algebraic theories free variables often range over existing objects only. However, we can formalise this as well:

Categories: "Algebraic reading" of axiom set by Freyd and Scedrov.

A1	$\forall xy.\ E(x \cdot y) \leftrightarrow dom\ x \cong cod\ y$
A2a	$\forall x.\ cod(dom\ x) \cong dom\ x$
A2b	$\forall y.\ dom(cod\ y) \cong cod\ y$
A3a	$\forall x.\ x \cdot (dom\ x) \cong x$
A3b	$\forall y.\ (cod\ y) \cdot y \cong y$
A4a	$\forall xy.\ dom(x \cdot y) \cong dom((dom\ x) \cdot y)$
A4b	$\forall xy.\ cod(x \cdot y) \cong cod(x \cdot (cod\ y))$
A5	$\forall xyz.\ x \cdot (y \cdot z) \cong (x \cdot y) \cdot z$

Experiments with Isabelle/HOL

But: Strictness is not mentioned in Freyd and Scedrov!
And it could not even be expressed axiomatically, when variables range over of existing objects only. This leaves us puzzled about their axiom system.

Hence, we better prefer the Axioms Set V by Scott (from 1977).

GROUPS, CATEGORIES AND DUALITY

BY SAUNDERS MACLANE*

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

Communicated by Marshall Stone, May 1, 1948

It has long been recognized that the theorems of group theory display a certain duality. The concept of a lattice gives a partial expression for this duality, in that some of the theorems about groups which can be formulated in terms of the lattice of subgroups of a group display the customary lattice duality between meet (intersection) and join (union). The duality is not always present, in the sense that the lattice dual of a
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introduced the notion of a category.[6] A *category* is a class of "mappings" (say, homomorphisms) in which the product $\alpha\beta$ of certain pairs of mappings α and β is defined. A mapping e is called an *identity* if $\rho\alpha = \alpha$ and $\beta\rho = \beta$ whenever the products in question are defined. These products must satisfy the axioms:

(C-1). *If the products $\gamma\beta$ and $(\gamma\beta)\alpha$ are defined, so is $\beta\alpha$;*
(C-1′). *If the products $\beta\alpha$ and $\gamma(\beta\alpha)$ are defined, so is $\gamma\beta$;*
(C-2). *If the products $\gamma\beta$ and $\beta\alpha$ are defined, so are the products $(\gamma\beta)\alpha$ and $\gamma(\beta\alpha)$, and these products are equal.*
(C-3). *For each γ there is an identity e_D such that γe_D is defined;*
(C-4). *For each γ there is an identity e_R such that $e_R\gamma$ is defined.*

It follows that the identities e_D and e_R are unique; they may be called, respectively, the *domain* and the *range* of the given mapping γ. A mapping θ with a two-sided inverse is an *equivalence*.

These axioms are clearly self dual, and a dual theory of free and direct products may be constructed in any category in which such products exist.

As before, we adopt an algebraic reading and add an explicit strictness condition.

Categories: Axioms Set by Mac Lane

C0	$E(\gamma \cdot \beta) \rightarrow (E\gamma \wedge E\beta)$	**(added by us)**
C1	$\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$	
C1'	$\forall \gamma, \beta, \alpha.\ (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \rightarrow E(\gamma \cdot \beta)$	
C2	$\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow$ $(E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$	
C3	$\forall \gamma.\ \exists eD.\ IDMcL(eD) \wedge E(\gamma \cdot eD)$	
C4	$\forall \gamma.\ \exists eR.\ IDMcL(eR) \wedge E(eR \cdot \gamma)$	

where $IDMcL(\rho) \equiv (\forall \alpha.\ E(\rho \cdot \alpha) \rightarrow \rho \cdot \alpha = \alpha) \wedge (\forall \beta.\ E(\beta \cdot \rho) \rightarrow \beta \cdot \rho = \beta)$

Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.

How about the Skolemized variant?

Categories: Axioms Set by Mac Lane

C0	$(E(\gamma \cdot \beta) \rightarrow (E\gamma \wedge E\beta)) \wedge (E(dom\ \gamma) \rightarrow (E\gamma)) \wedge (E(cod\ \gamma) \rightarrow (E\gamma))$	**(added)**
C1	$\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$	
C1'	$\forall \gamma, \beta, \alpha.\ (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \rightarrow E(\gamma \cdot \beta)$	
C2	$\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow (E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$	
C3	$\forall \gamma.\ IDMcL(dom\ \gamma) \wedge E(\gamma \cdot (dom\ \gamma))$	
C4	$\forall \gamma.\ IDMcL(cod\ \gamma) \wedge E((cod\ \gamma) \cdot \gamma)$	

Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **NITPICK**.

This axioms set is equivalent to (as shown by Sledgehammer)

Categories: Axioms Set V (Scott, 1977)

$S1$	Strictness	$E(dom\ x) \rightarrow Ex$
$S2$	Strictness	$E(cod\ y) \rightarrow Ey$
$S3$	Existence	$E(x \cdot y) \leftrightarrow dom\ x \simeq cod\ y$
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How about the Skolemized variant?

Categories: Axioms Set by Mac Lane

C0 $(E(\gamma \cdot \beta) \rightarrow (E\gamma \wedge E\beta)) \wedge (E(dom\ \gamma) \rightarrow (E\gamma)) \wedge (E(cod\ \gamma) \rightarrow (E\gamma))$ **(added)**

C1 $\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E((\gamma \cdot \beta) \cdot \alpha)) \rightarrow E(\beta \cdot \alpha)$

C1' $\forall \gamma, \beta, \alpha.\ (E(\beta \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \rightarrow E(\gamma \cdot \beta)$

C2 $\forall \gamma, \beta, \alpha.\ (E(\gamma \cdot \beta) \wedge E(\beta \cdot \alpha)) \rightarrow$
$(E((\gamma \cdot \beta) \cdot \alpha) \wedge E(\gamma \cdot (\beta \cdot \alpha)) \wedge ((\gamma \cdot \beta) \cdot \alpha) = (\gamma \cdot (\beta \cdot \alpha)))$

C3 $\forall \gamma.\ IDMcL(dom\ \gamma) \wedge E(\gamma \cdot (dom\ \gamma))$

C4 $\forall \gamma.\ IDMcL(cod\ \gamma) \wedge E((cod\ \gamma) \cdot \gamma)$

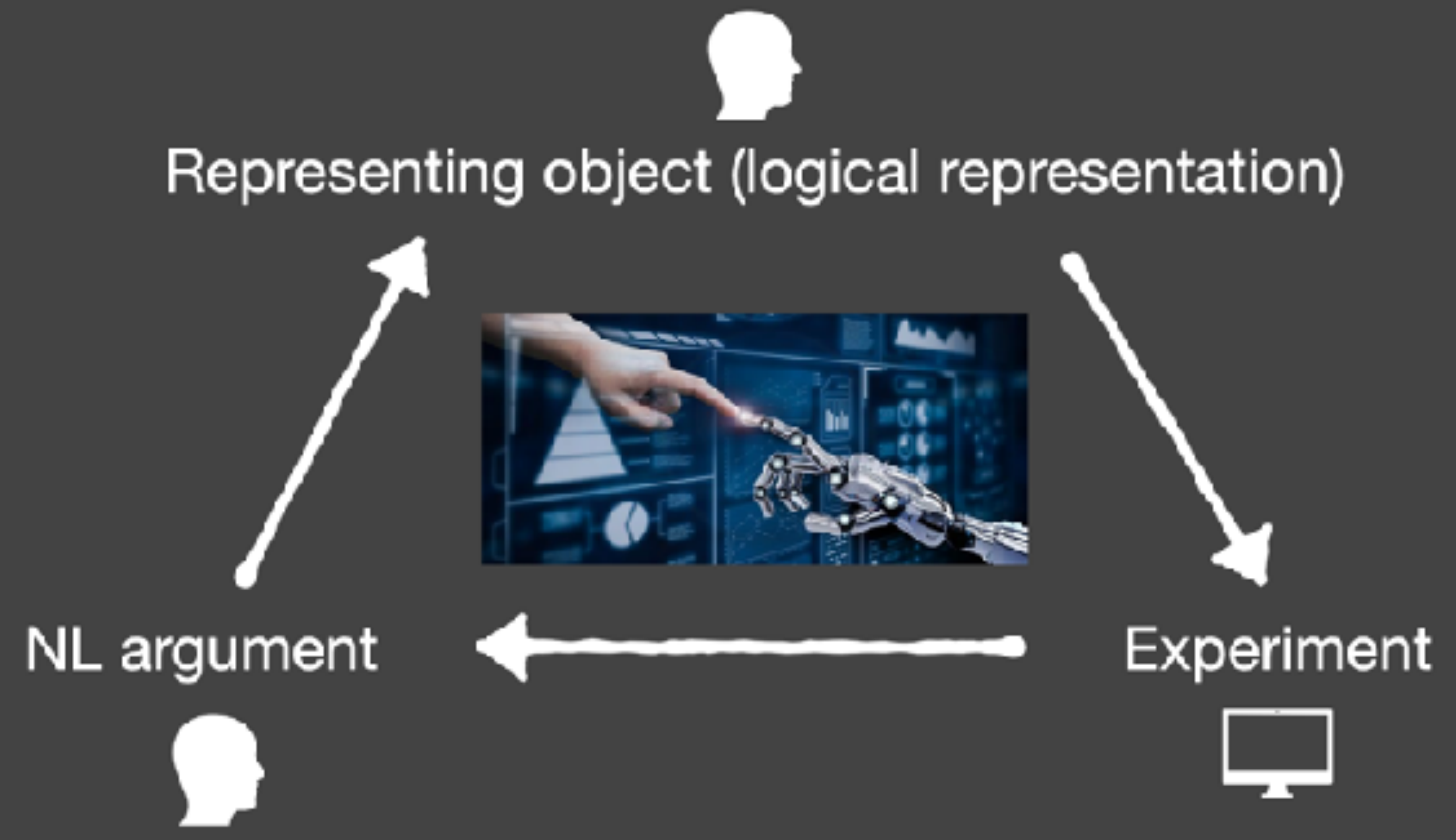
Consistency holds (also when $\exists x. \neg(Ex)$): confirmed by **Nitpick**.

See also our "Archive of Formal Proofs" entry at:
`https://www.isa-afp.org/entries/AxiomaticCategoryTheory.html`

Results of this study

Axiom Systems for Category Theory

- Connection depicted to generalised monoids
- Minimal axiom systems, dependencies
- Consistency, strictness assumptions
- Mutual relationships explored



Methological Results

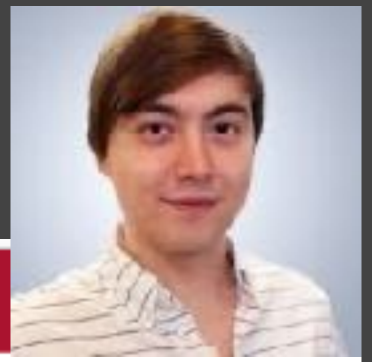
- Evidence for LogiKEy methodology
- High degree of automation: theorem proving & (counter-)model finding
- Required familiarity with Isabelle/HOL still (too) high for non-experts

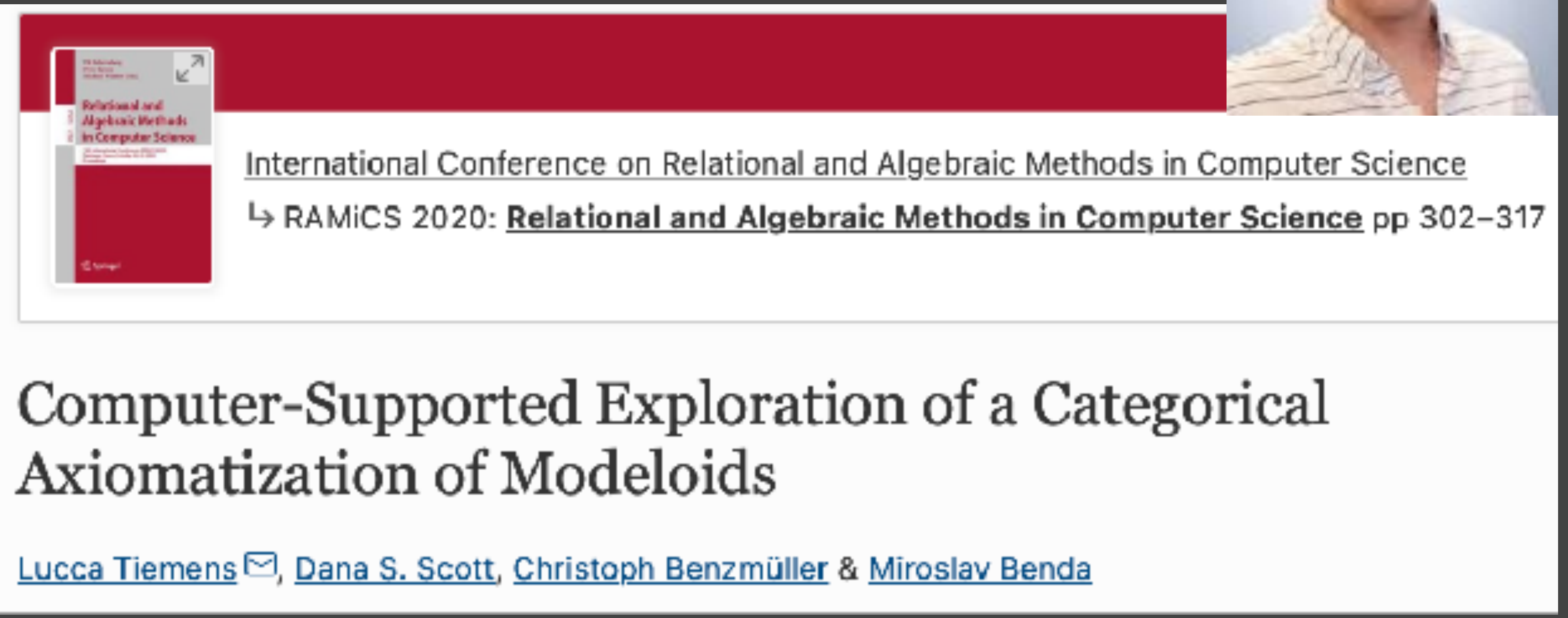
Obvious Question

- How about digging deeper?

Further Experiments

lucca@tiemens.de



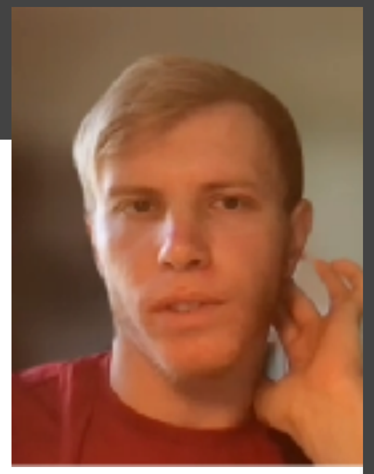


International Conference on Relational and Algebraic Methods in Computer Science
↳ RAMiCS 2020: Relational and Algebraic Methods in Computer Science pp 302–317

Computer-Supported Exploration of a Categorical Axiomatization of Modeloids

Lucca Tiemens, Dana S. Scott, Christoph Benzmüller & Miroslav Benda

gonus.aleksey@gmail.com



Freie Universität Berlin

Categorical semantics of Intuitionistic Multiplicative Linear Logic and its formalization in Isabelle/HOL

Master's thesis

17 May 2021

jonas.bayer@fu-berlin.de

Freie Universität Berlin

Bachelor's Thesis

Exploring categories, formally

2021

Jonas Bayer

A **modeloid** abstracts from a structure to the set of its partial automorphisms.

Using our axiomatisation of category theory we develop a generalization of a modeloid first to an inverse semigroup and then to an **inverse category**.

Formal framework to study relationship between structures of same vocabulary.

Abstract representation of Ehrenfeucht-Fraisse games between two structures.

Focuses on fragment of linear logic: **intuitionistic multiplicative LL (IMLL)**; further generalisation possible.

Using our axiomatisation of category theory an interpretation of IMLL formulas and rules in **symmetric monoidal closed categories** is presented.

Sound Modeling & Automation: IMLL modelled in Axiomatic Category Theory modelled in Free Logic modell. in HOL.

Studies **practicability/elegance** of axiomatic category theory approach.

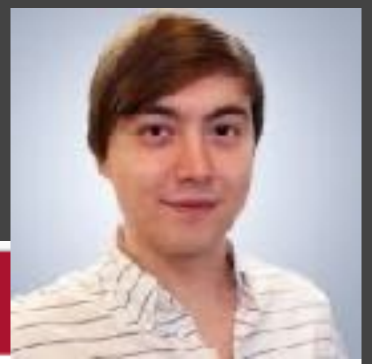
Studies **infinite structures:** category α-Set of functions between sets (with α-type elements); good automation.

Categ. with products & coproducts; some limitations discussed.

Category of categories: proves that categories themselves form a category with functors as arrows.

Further Experiments

lucca@tiemens.de



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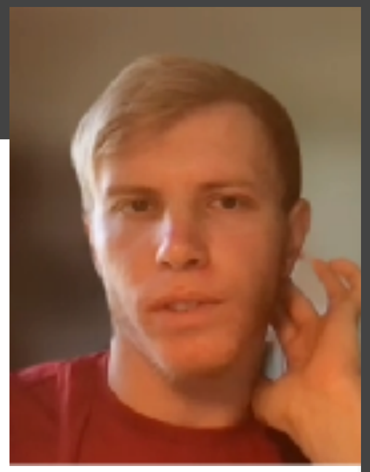

Freie Universität Berlin

Categorical semantics of Intuitionistic Multiplicative Linear Logic and its formalization in Isabelle/HOL

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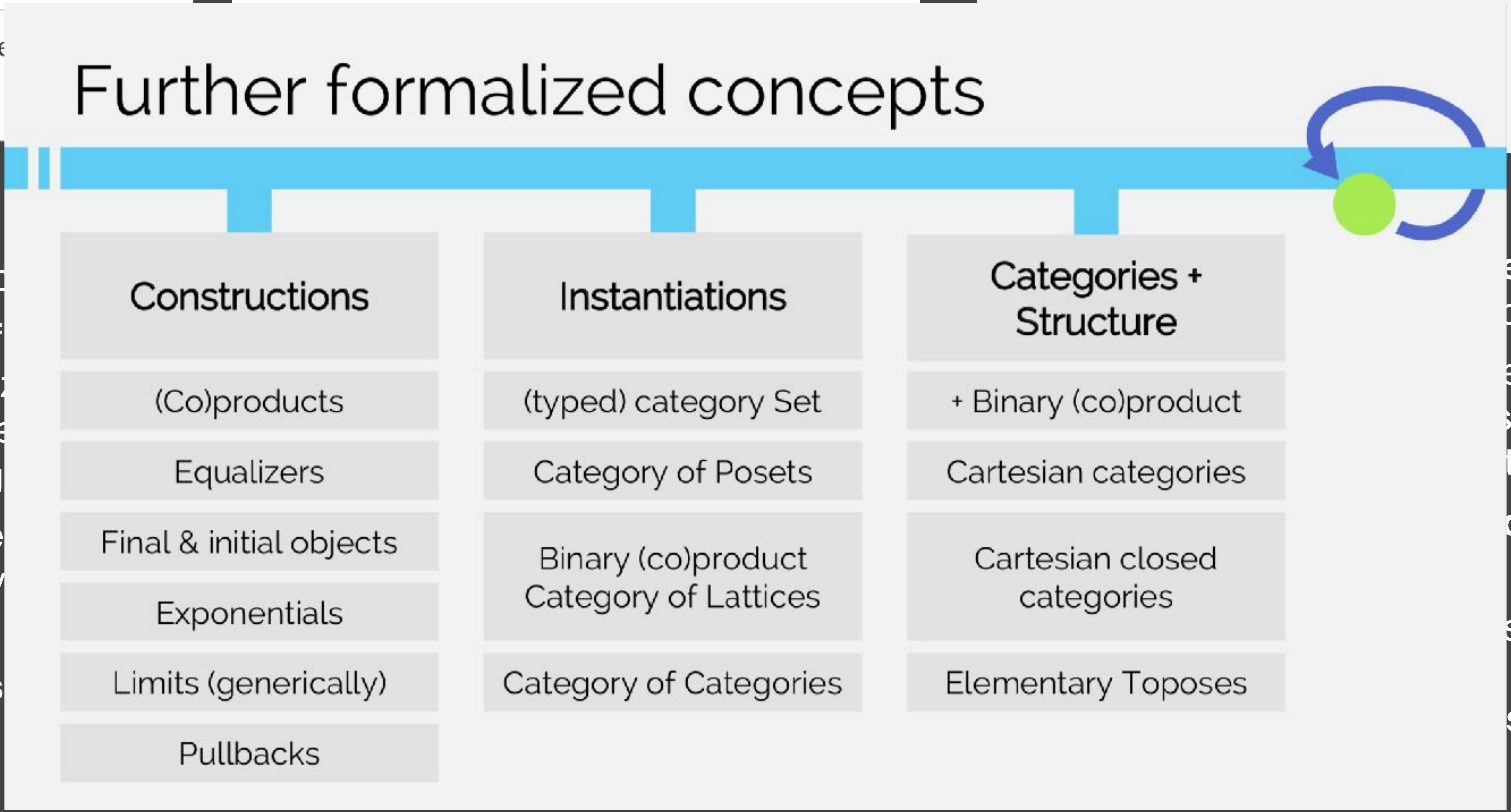
FREIE UNIVERSITÄT BERLIN

BACHELOR'S THESIS

Exploring categories, formally



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Further Foundational Studies: Metaphysical Theory

(PhD of Daniel Kirchner, supervised by Ed Zalta and myself)

Principia Logico-Metaphysica
(Draft/Excerpt)

Edward N. Zalta
Philosophy Department
Stanford University

With critical theoretical contributions by

Daniel Kirchner
Institut für Mathematik
Freie Universität Berlin
and
Uri Nodelman
Philosophy Department
Stanford University

October 13, 2021

http://mally.stanford.edu/principia.pdf

Computer-Verified Foundations of Metaphysics
and an Ontology of Natural Numbers in Isabelle/HOL

Dissertation
zur Erlangung des Grades eines
Doktors der Naturwissenschaften

am Fachbereich Mathematik und Informatik
der Freien Universität Berlin

vorgelegt von

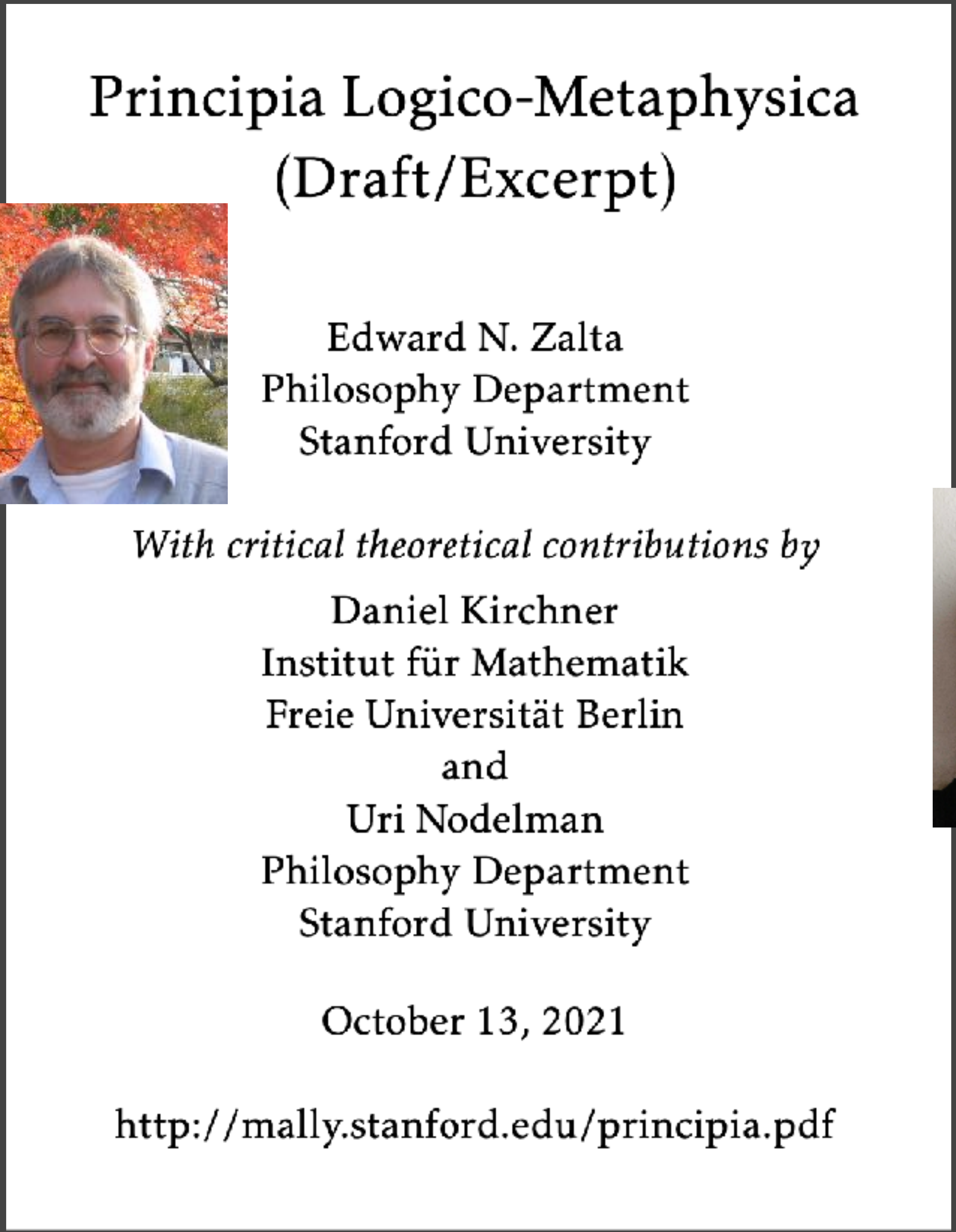
Daniel Kirchner

Berlin, December 2021

Entire PhD thesis was written directly in Isabelle/HOL

Further Foundational Studies: Metaphysical Theory

(PhD of Daniel Kirchner, supervised by Ed Zalta and myself)



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October 13, 2021

http://mally.stanford.edu/principia.pdf

Foundational metaphysical theory (based on a hyperintensional relational HO modal logic)

Formalised & studied in Isabelle/HOL

- approx. 24000 loc
- using LogiKEy methodology
- paradox rediscovered & fixed
- derivation of natural numbers

Latest versions of this theory shifted towards free logic; strongly influenced (& verified) by computer-experiments

Further Foundational Studies: Metaphysical Theory

(PhD of Daniel Kirchner, supervised by Ed Zalta and myself)

Principia Logico-Metaphysica
(Draft/Excerpt)

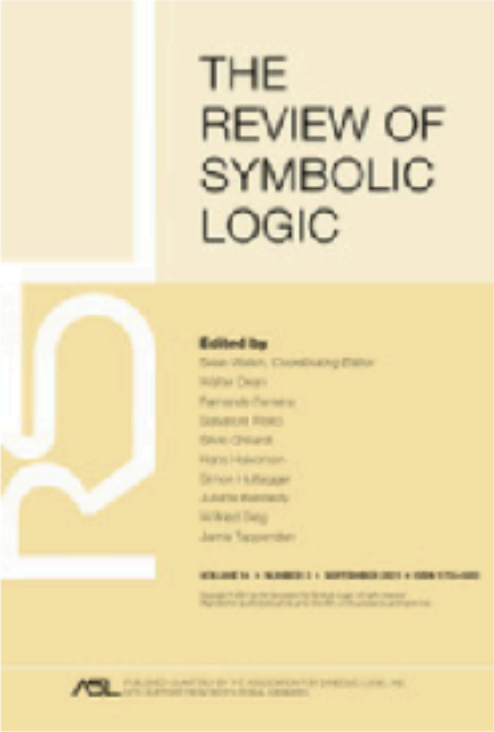
Edward N. Zalta
Philosophy Department
Stanford University

With critical theoretical contributions by

Daniel Kirchner
Institut für Mathematik
Freie Universität Berlin
and
Uri Nodelman
Philosophy Department
Stanford University

October 13, 2021

http://mally.stanford.edu/principia.pdf



MECHANIZING *PRINCIPIA LOGICO-METAPHYSICA* IN FUNCTIONAL TYPE-THEORY

Published online by Cambridge University Press: **12 July 2019**

DANIEL KIRCHNER, CHRISTOPH BENZMÜLLER and EDWARD N. ZALTA

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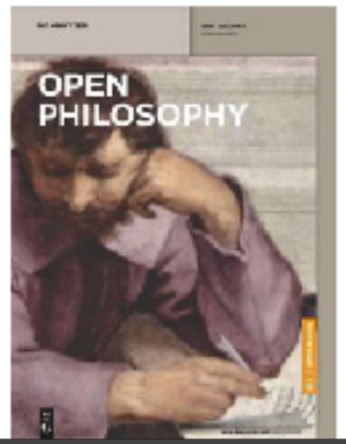


ARTICLE Open Access

Computer Science and Metaphysics: A Cross-Fertilization

Open Philosophy

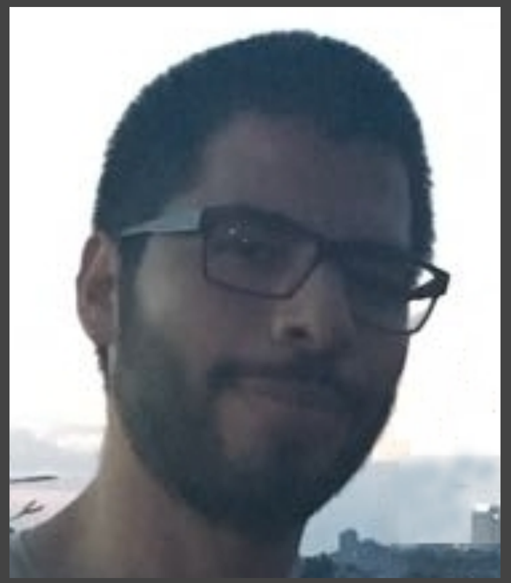
Daniel Kirchner, Christoph Benzmüller, Edward N. Zalta August 23, 2019

Isabelle/HOL Code (~24000 loc):
https://github.com/ekpyron/AOT

daniel@ekpyron.org

Further Foundational Studies: Topology

(ongoing PhD studies of David Fuenmayor)



Isabelle AFP

TOPOLOGICAL SEMANTICS FOR PARACONSISTENT AND PARACOMPLETE LOGICS

Title:	Topological semantics for paraconsistent and paracomplete logics
Author:	David Fuenmayor (davfuenmayor /at/ gmail /dot/ com)
Submission date:	2020-12-17
Abstract:	We introduce a generalized topological semantics for paraconsistent and paracomplete logics by drawing upon early works on topological Boolean algebras (cf. works by Kuratowski, Zarycki, McKinsey & Tarski, etc.). In particular, this work exemplarily illustrates the shallow semantical embeddings approach (SSE) employing the proof assistant Isabelle/HOL. By means of the SSE technique we can effectively harness theorem provers, model finders and 'hammers' for reasoning with quantified non-classical logics.
BibTeX:	`@article{Topological_Semantics-AFP,` [...]
License:	BSD License

Formalising Basic Topology for Computational Logic in Simple Type Theory

David Fuenmayor[1,2] and Fabian Serrano[3]

[1] University of Luxembourg, Esch-sur-Alzette, Luxembourg
[2] University of Bamberg, Bamberg, Germany
[3] National University of Colombia, Manizales, Colombia

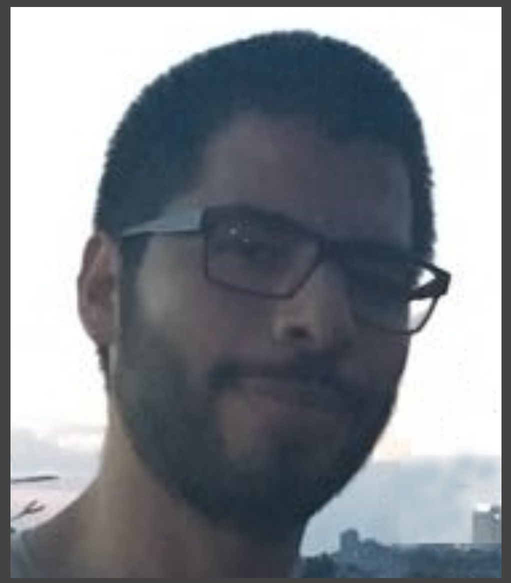
Abstract. We present a formalisation of basic topology in simple type theory encoded using the Isabelle/HOL proof assistant. In contrast to related formalisation work, which follows more 'traditional' approaches, our work builds upon closure algebras, encoded as Boolean algebras of (characteristic functions of) sets featuring an axiomatised closure operator (cf. seminal work by Kuratowski and McKinsey & Tarski). With this approach we primarily address students of computational logic, for whom we bring a different focus, closer to lattice theory and logic than to set theory or analysis. This approach has allowed us to better leverage the automated tools integrated into Isabelle/HOL (model finder Nitpick and Sledgehammer) to do most of the proof and refutation heavy-lifting, thus allowing for assumption-minimality and less-verbose interactive proofs.

Keywords: Closure Algebras · Topology · Simple Type Theory · Isabelle/HOL

https://github.com/davfuenmayor/basic-topology.git

Further Foundational Studies: Ethics

(ongoing PhD studies of David Fuenmayor)



Springer Link

Pacific Rim International Conference on Artificial Intelligence

↳ PRICAI 2019: **PRICAI 2019: Trends in Artificial Intelligence** pp 418–432 | Cite as

Harnessing Higher-Order (Meta-)Logic to Represent and Reason with Complex Ethical Theories

David Fuenmayor & Christoph Benzmüller

Normative Reasoning with Expressive Logic Combinations

Authors	David Fuenmayor, Christoph Benzmüller
Pages	2903 - 2904
DOI	10.3233/FAIA200445
Category	Research Article
Series	Frontiers in Artificial Intelligence and Applications
Ebook	Volume 325: ECAI 2020

Deontological ethical theory — **PGC by Alan Gewirth**

Isabelle AFP

Formalisation and Evaluation of Alan Gewirth's Proof for the Principle of Generic Consistency in Isabelle/HOL

- Home
- About
- Submission
- Updating Entries
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- Search
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Title:	Formalisation and Evaluation of Alan Gewirth's Proof for the Principle of Generic Consistency in Isabelle/HOL
Authors:	David Fuenmayor (davfuenmayor /at/ gmail /dot/ com) and Christoph Benzmüller
Submission date:	2018-10-30
Abstract:	An ambitious ethical theory ---Alan Gewirth's "Principle of Generic Consistency"--- is encoded and analysed in Isabelle/HOL. Gewirth's theory has stirred much attention in philosophy and ethics and has been proposed as a potential means to bound the impact of artificial general intelligence.
Change history:	[2019-04-09]: added proof for a stronger variant of the PGC and examplary inferences (revision 88182cb0a2f6)
BibTeX:	`@article{GewirthPGCProof-AFP, [...]`
License:	BSD License

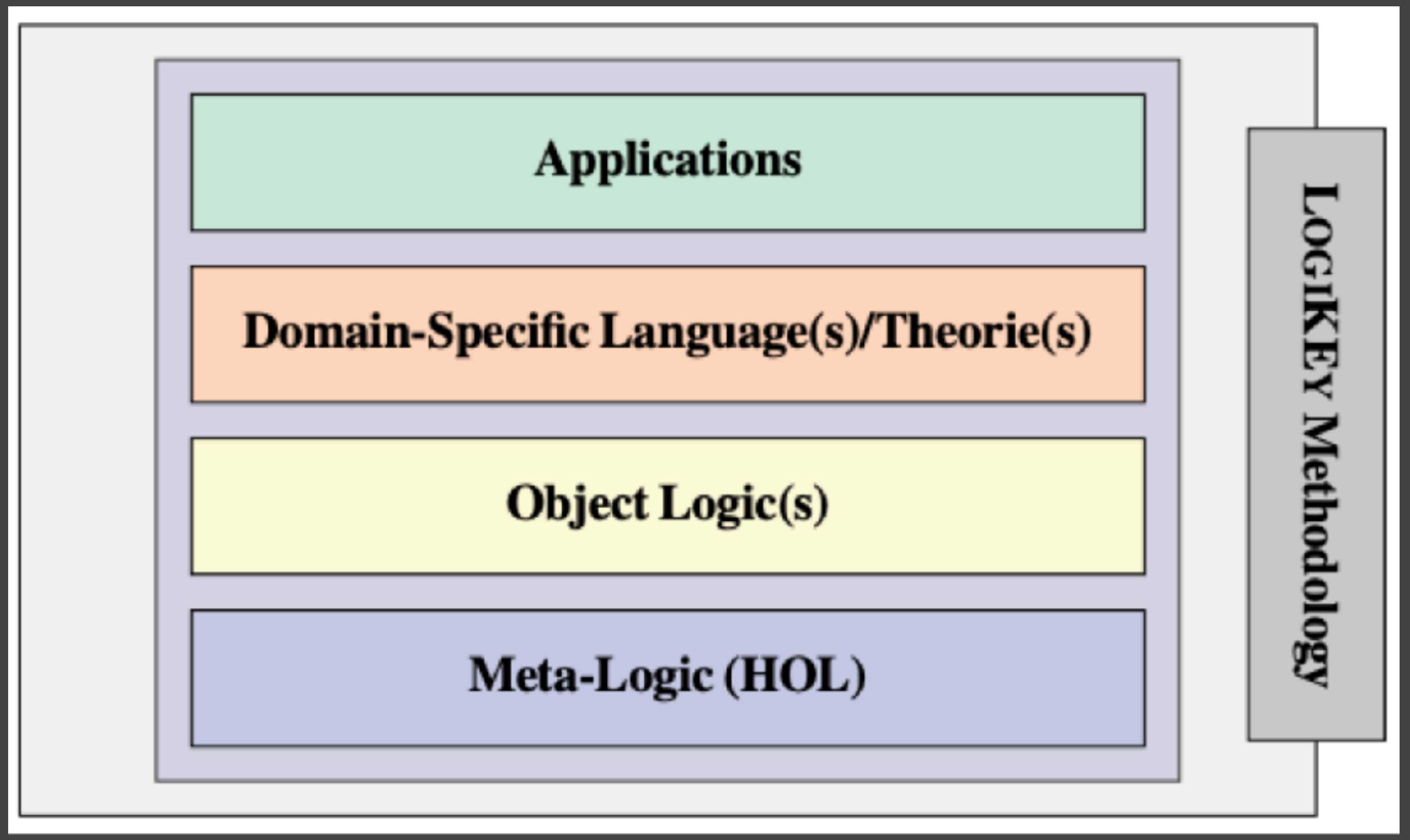
Our encoding utilises the **dyadic deontic logic** by Carmo and Jones as object logic; in fact, it uses a rich logic combination.

Conclusion: Successful Application(s) of LogiKEy

Human-Computer Interaction

Revision (often small changes):

Representing object (logical representation)



Modified experiments:

Revision:

Argument/Theory

Experiment

New insights (e.g. falsification)

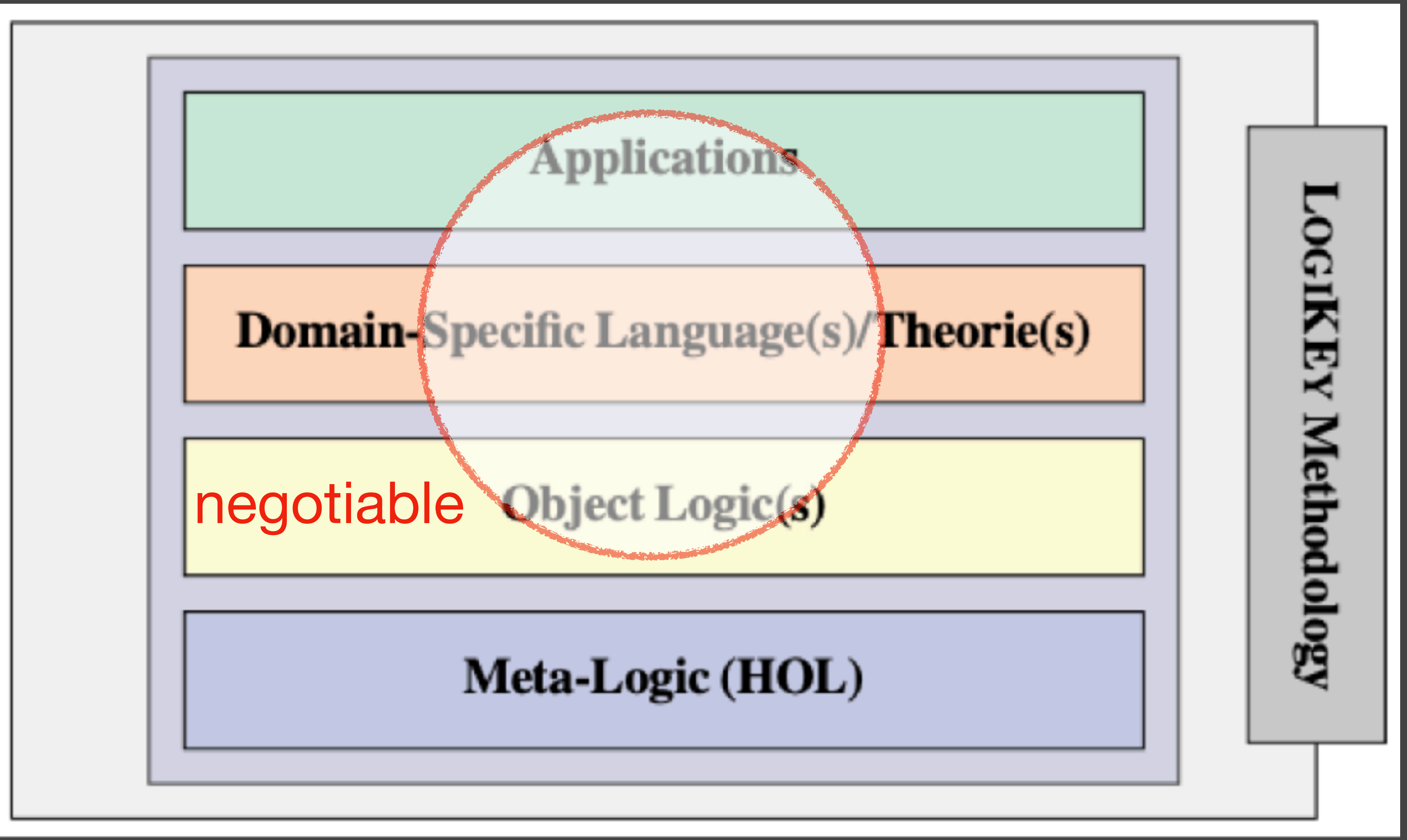
Conclusion: Logico-Pluralistic LogiKEy Approach

LogiKEy successfully applied for

- a wide range of object logics
- various object logic combinations
- different application domains (with contribution of new insights)

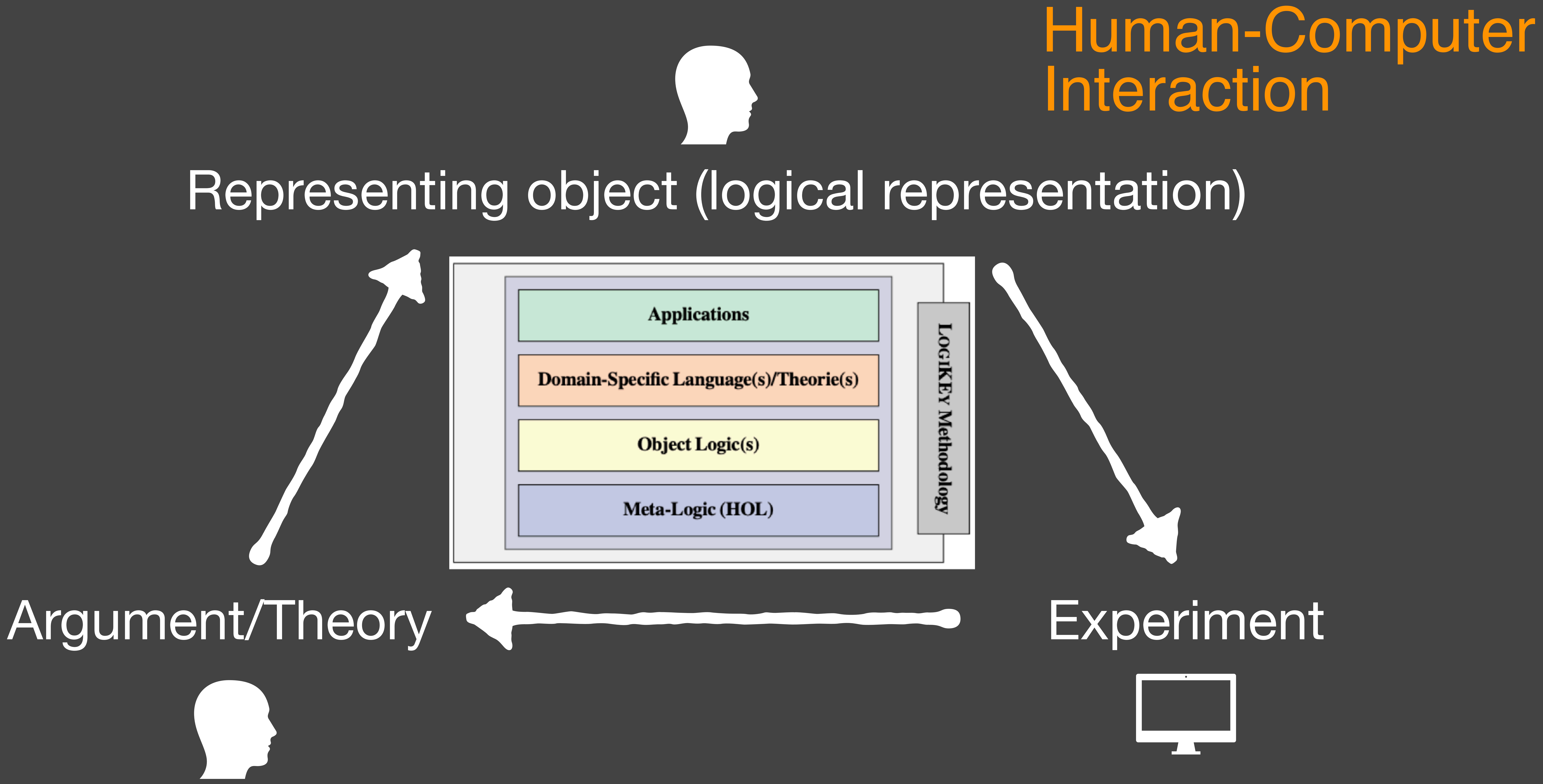
LogiKEy in Isabelle/HOL

- good proof automation with Sledgehammer
- even more valuable is (counter-)model finding with Nitpick
- very good syntax representations



LogiKEy offers a uniform methodology and infrastructure where even object logics and their conbinations become negotiable and objects of study.

Conclusion: Successful Application(s) of LogiKEy

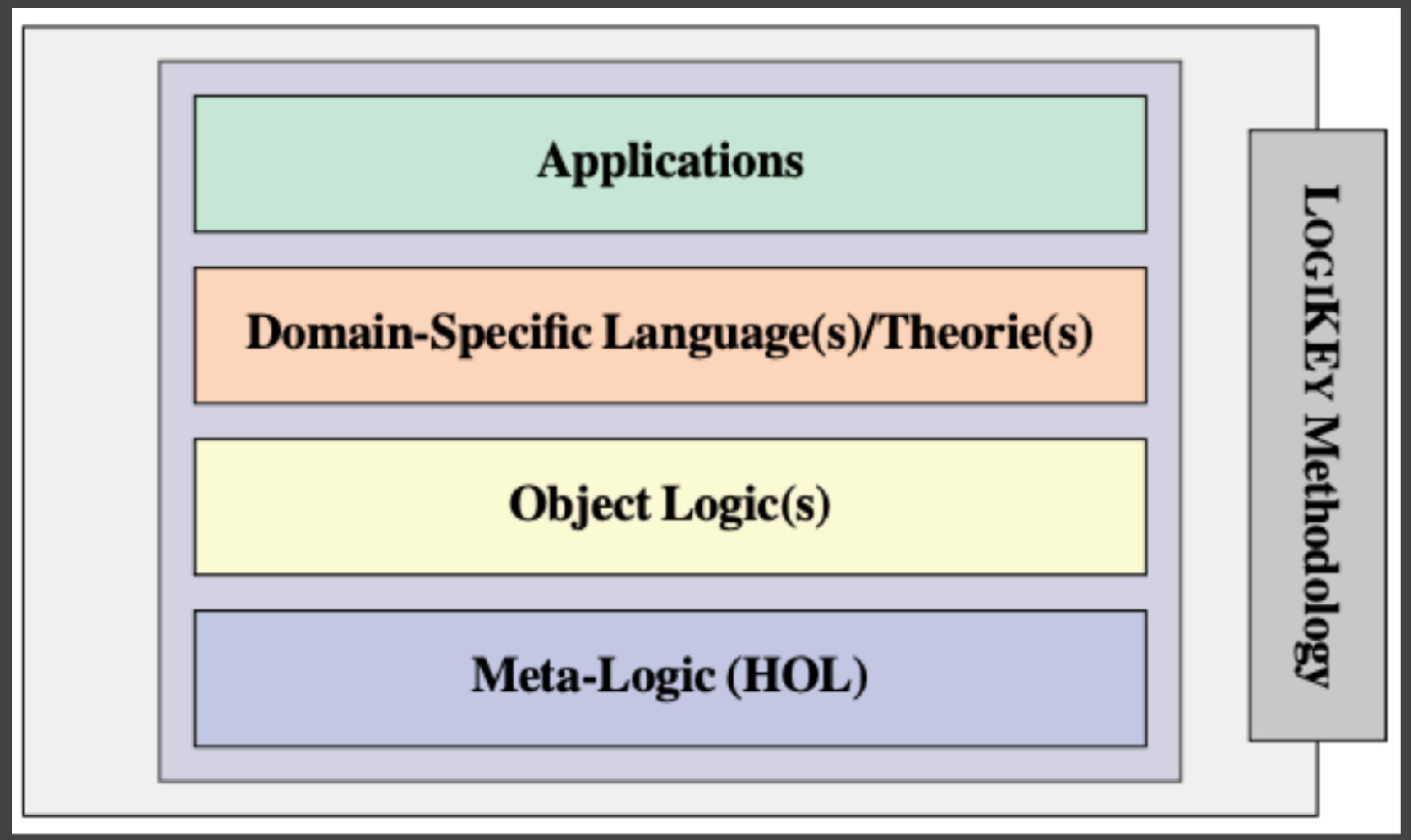


Conclusion: Successful Application(s) of LogiKEy

This is not (yet)
what we are doing!

Argument/Theory

Representing object (logical representation)



But maybe it's
doable to some
degree?

Argument/Theory

Experiment

Conclusion: Maths, Metaphysics & Experimental Sciences

> … The difference is that the natural scientists base their answers on observation, experiment, measurement and calculation, while the metaphysicians base theirs on armchair reflection …

(Timothy Williamson, Oxford, in an article for the British Academy, 14 Aug 2020)

This seems not completely true anymore.

The differences between metaphysics, maths and experimental sciences could gradually disappear?

But clearly: **Representing Objects, Logic** & **Symbolic AI** are needed.

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