Categorification of negative information using enrichment

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- **What is negative information, and why do we care?**
  - It pops up in practical applications, e.g., infeasibility results in robot motion planning.
  - We asked: **what is the corresponding categorical notion?**
- **Idea:** represent negative information by negative arrows called "**norphisms**," which complement the positive information of **morphisms.**
- A **nategory** is a category with some additional structure for norphisms accounting,
- Norphisms do not compose by themselves. They need a morphism as a "catalyst."

$$\frac{f : X \to Y \quad g : Y \to Z}{(f \mathbin{;} g) : X \to Z} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!- n}{\dashrightarrow} Z} \qquad \frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n -\!\bullet g}{\dashrightarrow} Y}$$

- We can derive the norphism rules very elegantly using **enriched category theory.**
  - Just like a $\mathbf{P} := \langle \mathbf{Set}, \times, 1 \rangle$ -enriched category provides the data for a small category, …
  - … a **PN**-enriched category provides the data for a nategory, where **PN** is a category based on De Paiva's **GC** construction.
- **Conclusions**: morphisms and norphisms are of the same substance. **Negative information can be "categorified" using enriched category theory.**

Example: robot motion planning

- **Robot motion planning:** find the optimal path between two robot configurations. Paths should avoid obstacles and have a cost (e.g., fuel required, minimum time).

- As a category: objects are points in "free space," and morphisms are paths with a cost. Morphism composition concatenates the paths and "sums" the costs.

Example: Dijkstra’s algorithm

- **Dijkstra’s algorithm** searches a path from start to goal that minimizes the traversal cost.
- Exploration is **uninformed**.

node	priority	CTC	CTG
start	0.00	0.00	0.00

A* ("A star")

- **A*** searches a path from start to goal that minimizes the traversal cost.
- Exploration is **informed**:
  - we have a **heuristic**: a **lower bound** on the cost-to-go from a node to the target.

node	priority	CTC	CTG
start	9.00	0.00	9.00

Example: robot motion planning

- **Robot motion planning:** find the optimal path between two robot configurations. Paths should avoid obstacles and have a cost (e.g., fuel required, minimum time).

- As a category: objects are points in "free space," and morphisms are paths with a cost. Morphism composition concatenates the paths and "sums" the costs.

- A ***complete*** algorithm can find a **path** (if it exists) or give a **certificate of infeasibility** (if one doesn't exist).

positive information: morphism!

what is this, categorically?

- An ***optimal*** algorithm can find (if it exists) an **optimal solution**:
  - **a feasible path,** plus...
  - **a certificate of optimality**: there is no better path.

positive information: morphism!

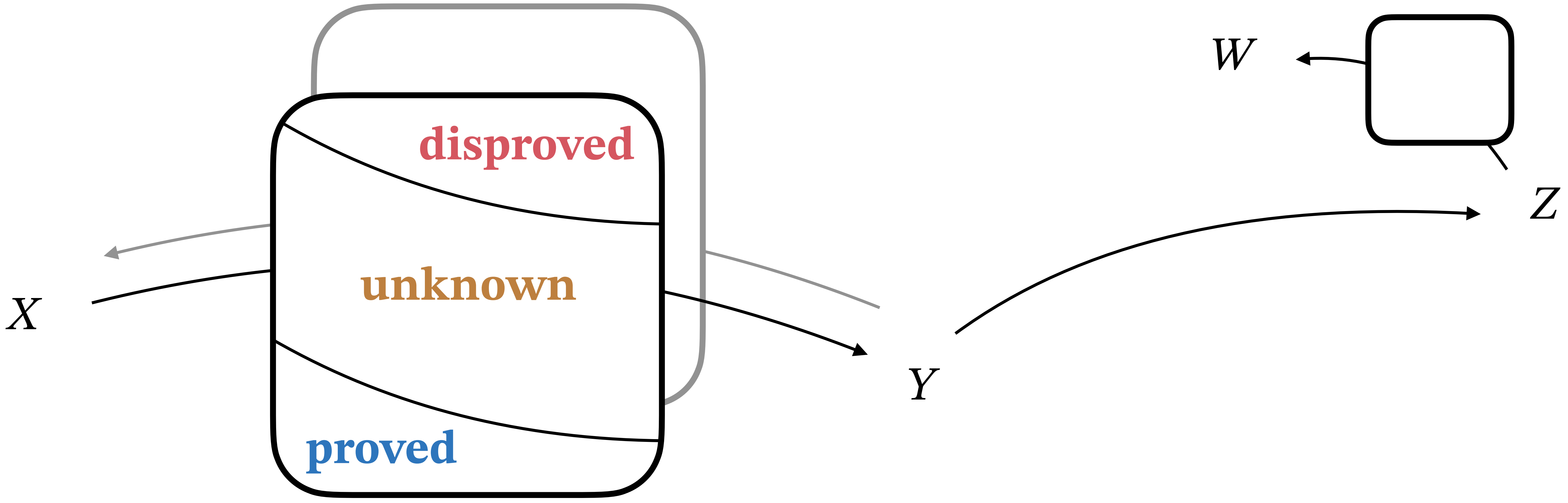
what is this, categorically?

- Search algorithms of the A* family achieve speed using **heuristics: lower bounds for the cost between** two points.

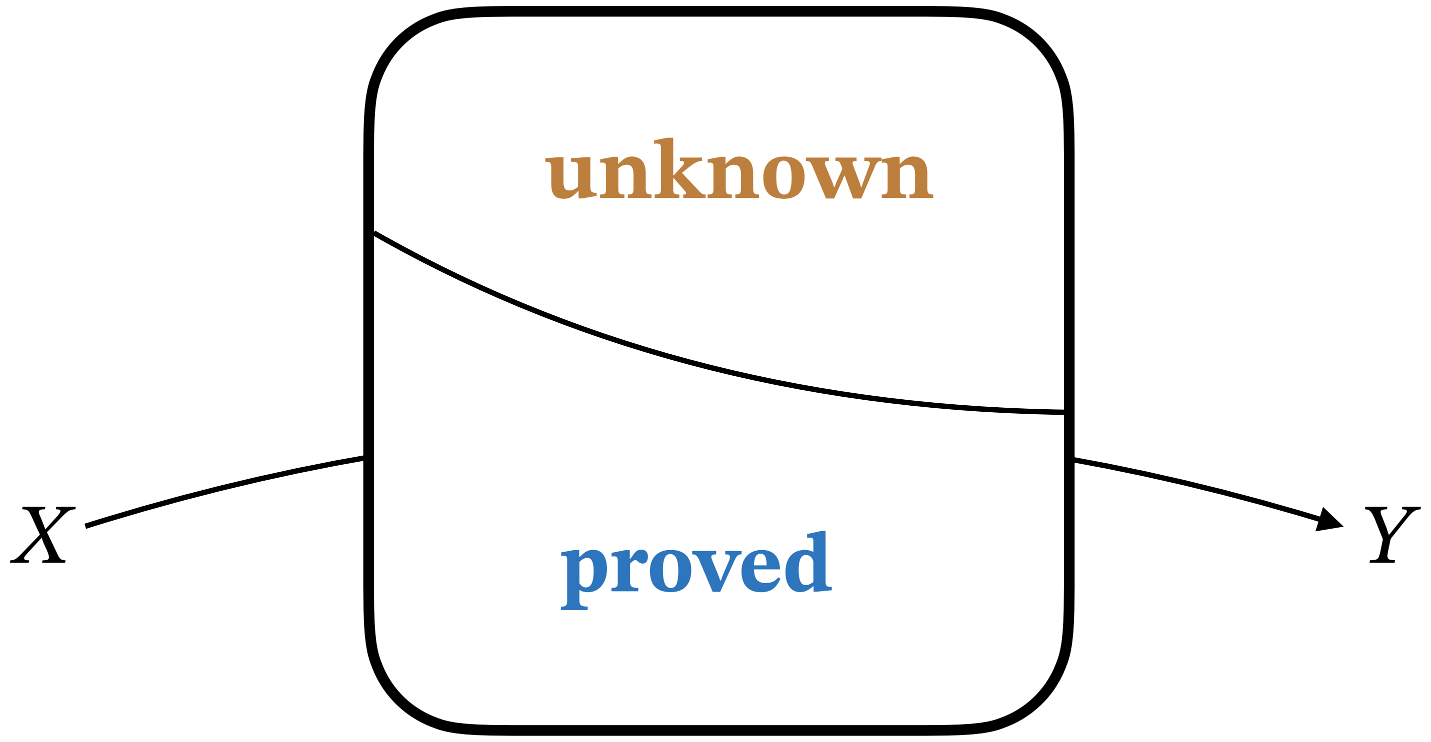
what is this, categorically?

Absence of evidence *vs* evidence of absence

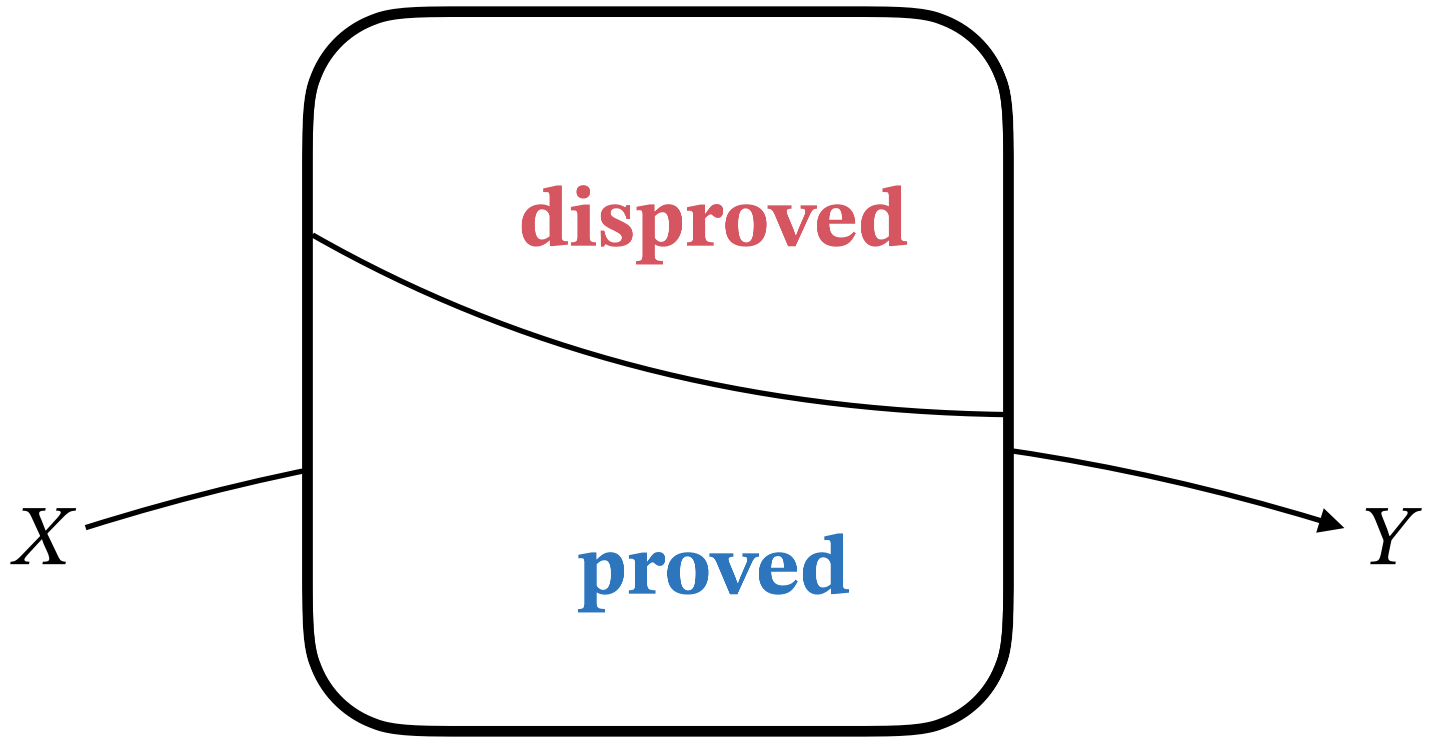
- More in general, it is common to have algorithms that run some kind of “inference” procedure that produces “feasible points” (morphisms).
- At each instant, each morphism is either “**proved**”, “**disproved**”, or “**unknown**”.



absence of evidence



evidence of absence



Building intuition: the case of thin categories

- In a thin category, there is at most one morphism per hom-set.
- These are preorders that represent connectivity. (Motion planning without costs.)

- **We postulate these semantics:**
  - A norphisms $n : X \dashrightarrow Y$ implies that there is no morphism $f : X \to Y$
  - A morphism $f : X \to Y$ implies that there is no norphism $n : X \dashrightarrow Y$

- We find that **the norphisms rules are dual to the morphisms rules**

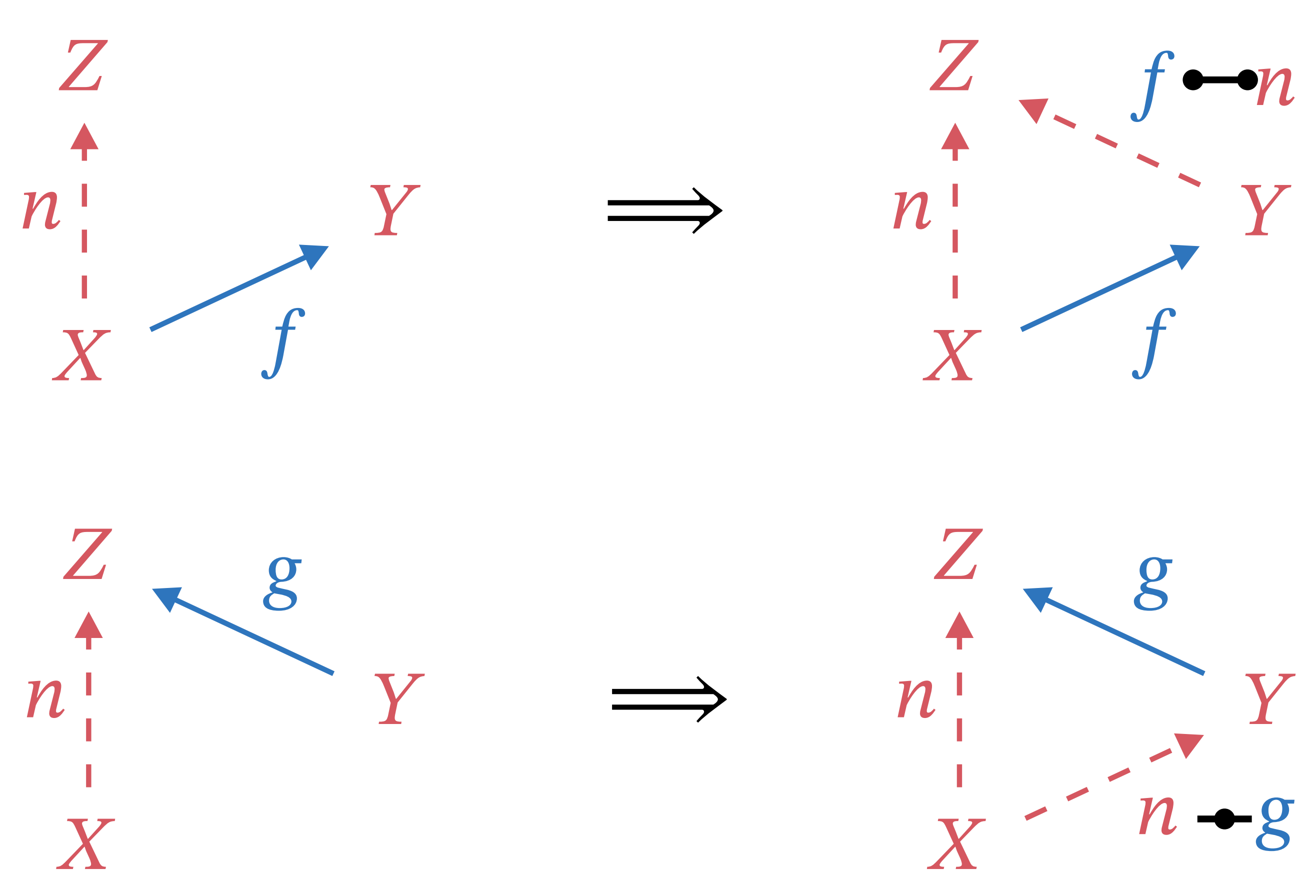
$$\frac{\top}{X \to X} \qquad \frac{f : X \to Y \quad g : Y \to Z}{(f \mathbin{;} g) : X \to Z}.$$

$$\frac{X \dashrightarrow X}{\bot} \qquad \frac{o : X \dashrightarrow Z \qquad Y : \mathrm{Ob}_{\mathbf{C}}}{(n : X \dashrightarrow Y) \vee (m : Y \dashrightarrow Z)}.$$

Note: nonconstructive!

Norphisms composition needs morphisms as catalysts

- We **constructively** revisit the logic to obtain **composition rules**.
- The constraint splits into **two rules** of the type morphism + norphism → norphism:



$$\frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \multimap n}{\dashrightarrow} Z}$$

$$\frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n \multimapinv g}{\dashrightarrow} Y}$$

- Norphism composition requires morphisms as catalysts.
- There is no norphism + norphism composition rule.

$$\frac{n : X \dashrightarrow Y \quad m : Y \dashrightarrow Z}{??? : X \dashrightarrow Z}$$

- There is no "category of norphisms."
- Norphisms are complementary to morphisms but obey different rules.

Definition (Nategory)

A locally small *nategory* $\mathbf{C}$ is a locally small category with the following additional structure. For each pair of objects $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, in addition to the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(X;Y)$, we also specify:

- A set of norphisms $\mathrm{Nom}_{\mathbf{C}}(X;Y)$.
- An *incompatibility relation*, which we write as a binary function

$$i_{XY} : \mathrm{Nom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(X;Y) \to \mathbf{Bool}.$$

For all triples X, Y, Z, in addition to the morphism composition function

$$\fatsemi_{XYZ} : \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Hom}_{\mathbf{C}}(X;Z),$$

we require the existence of two norphism composition functions

$$\begin{aligned} \bullet\!\!-\!\!\bullet_{XYZ} &: \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Nom}_{\mathbf{C}}(X;Z) \to \mathrm{Nom}_{\mathbf{C}}(Y;Z), \\ -\!\!\bullet_{XYZ} &: \mathrm{Nom}_{\mathbf{C}}(X;Z) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Nom}_{\mathbf{C}}(X;Y), \end{aligned}$$

and we ask that they satisfy two "equivariance" conditions:

$$i_{YZ}(f \bullet\!\!-\!\!\bullet\, n, g) \Rightarrow i_{XZ}(n, f \fatsemi g), \tag{equiv-1}$$

$$i_{XY}(n -\!\!\bullet\, g, f) \Rightarrow i_{XZ}(n, f \fatsemi g). \tag{equiv-2}$$

- We call a nategory "*exact*" if:

$$\begin{aligned} i_{YZ}(f \bullet\!\!-\!\!\bullet\, n, g) &\Leftrightarrow i_{XZ}(n, f \fatsemi g) \\ i_{XY}(n -\!\!\bullet\, g, f) &\Leftrightarrow i_{XZ}(n, f \fatsemi g) \end{aligned}$$

Canonical nategory constructions

- Here are some ways to get a nategory from a category **C**.

No norphisms

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) := \emptyset$$

One norphism

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) := \{\bullet\}$$

$$i_{XX}(\bullet, \mathrm{id}_X) = \bot$$

$$i_{XY}(\bullet, f) = \top$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} \bullet = \bullet$$

$$\bullet \mathbin{-\!\!\bullet\!\!-} g = \bullet$$

(for semicats)

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) := \{\bullet\}$$

~~$i_{XX}(\bullet, \mathrm{id}_X) = \bot$~~

$$i_{XY}(\bullet, f) = \top$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} \bullet = \bullet$$

$$\bullet \mathbin{-\!\!\bullet\!\!-} g = \bullet$$

The combinatorial explosion

$$\mathrm{Nom}_{\mathbf{C}}(X;Y) = \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y))$$

$$i_{XY}(n, f) = f \in n$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} n = \mathrm{pre}_f^{-1}(n)$$

$$n \mathbin{-\!\!\bullet\!\!-} g = \mathrm{post}_g^{-1}(n)$$

… with very weak inference rules

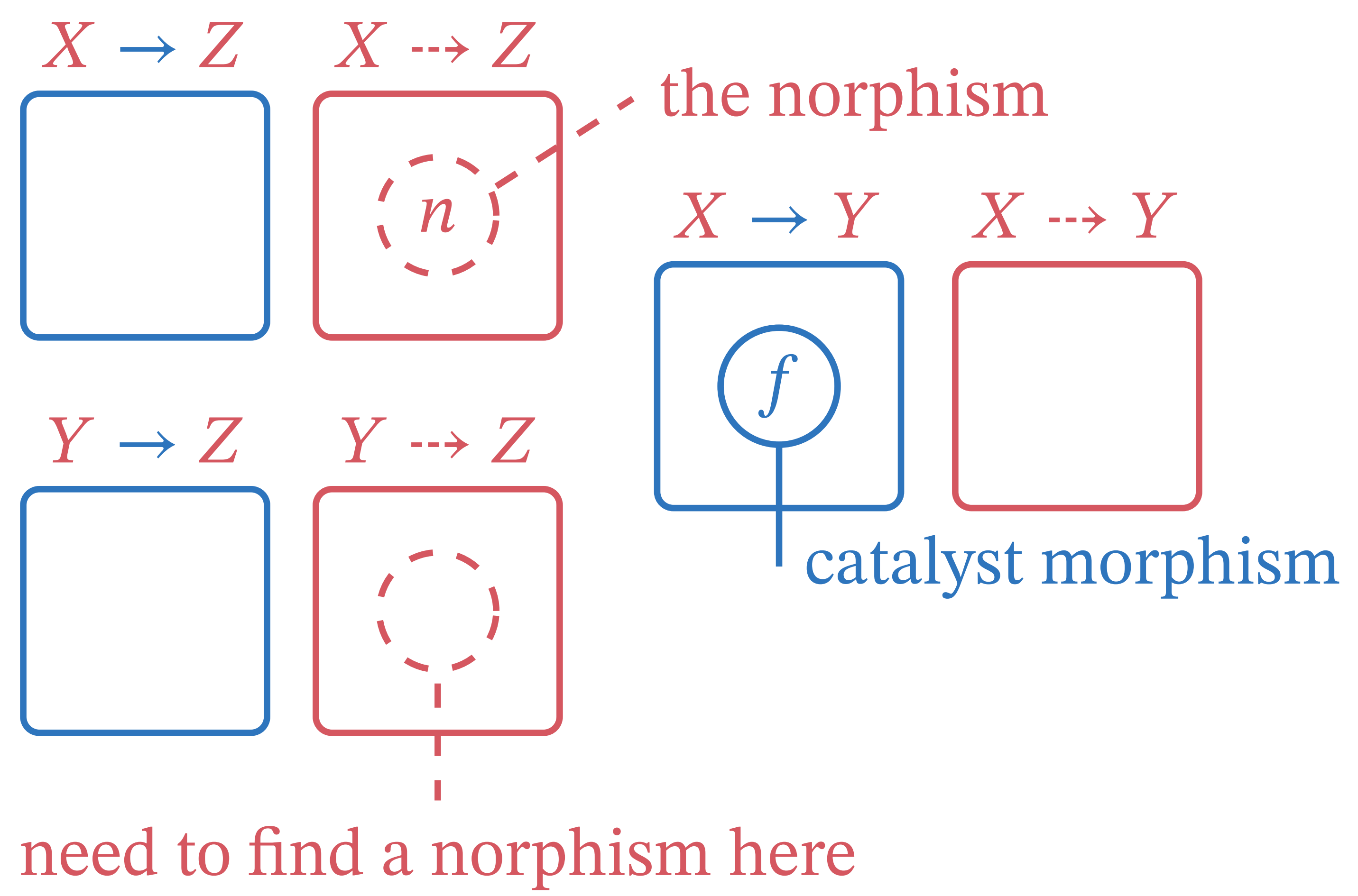
$$\mathrm{Nom}_{\mathbf{C}}(X;Y) = \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y))$$

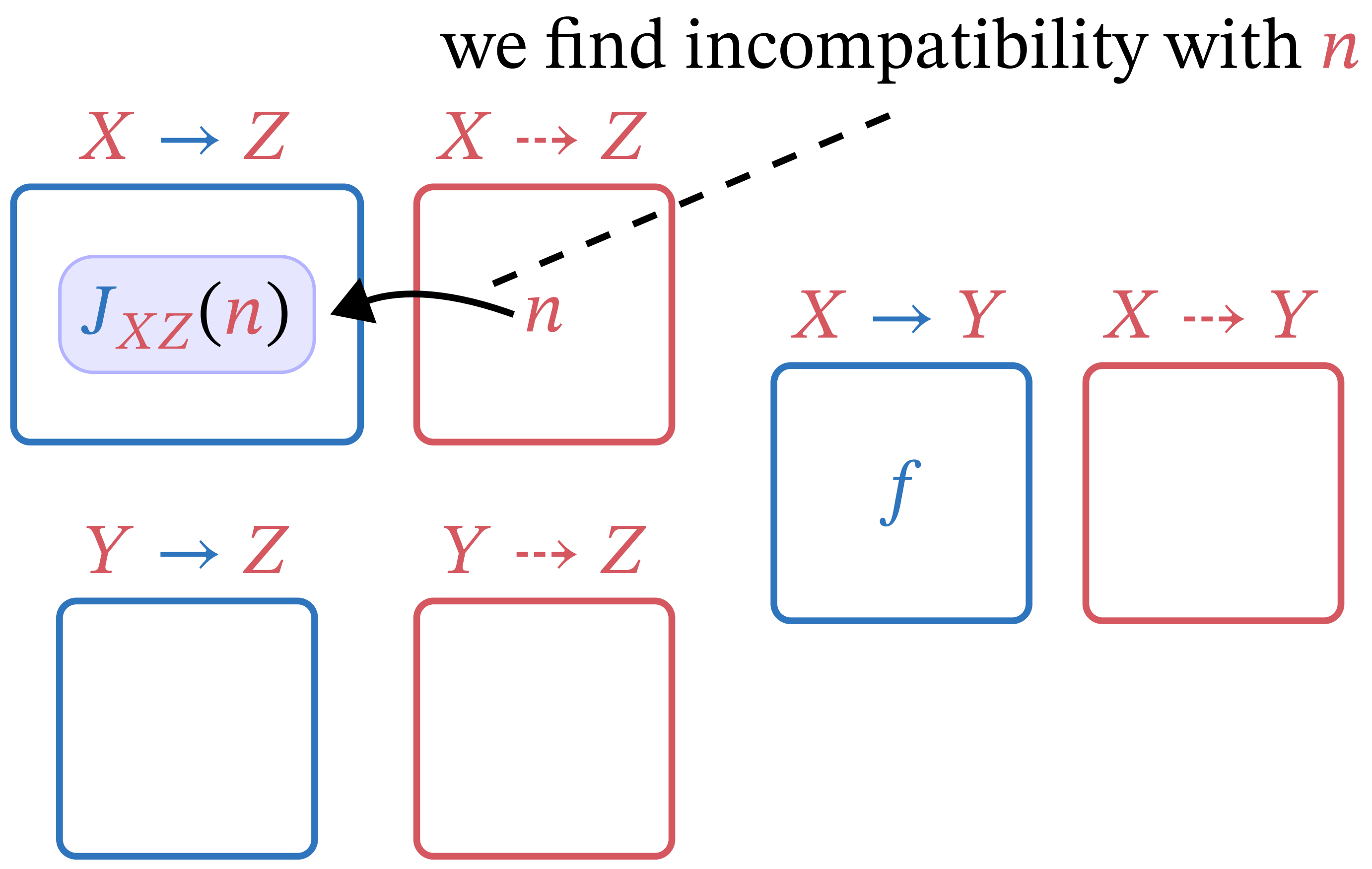
$$i_{XY}(n, f) = f \in n$$

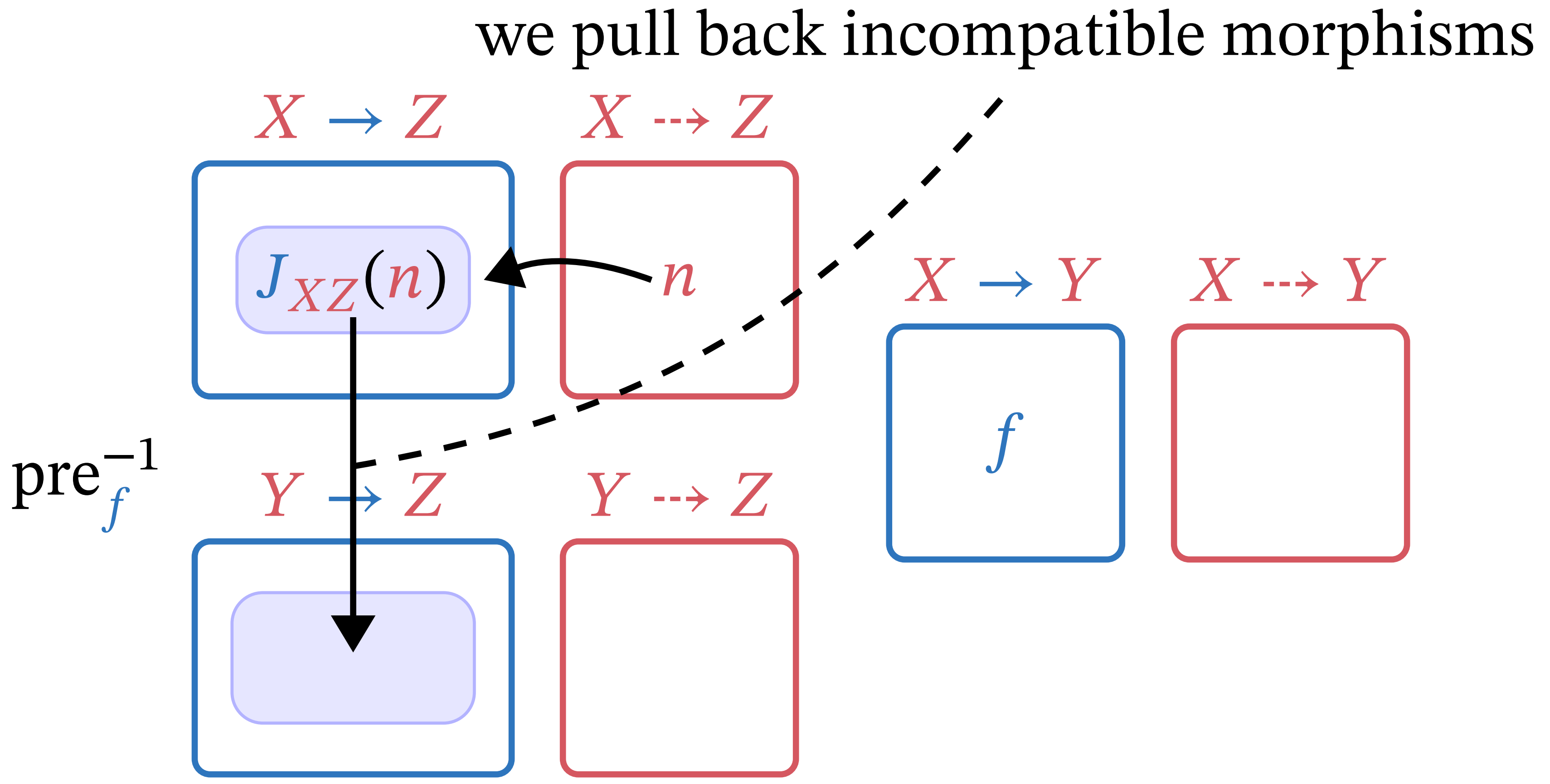
$$f \mathbin{\bullet\!\!-\!\!\bullet} n = \emptyset$$

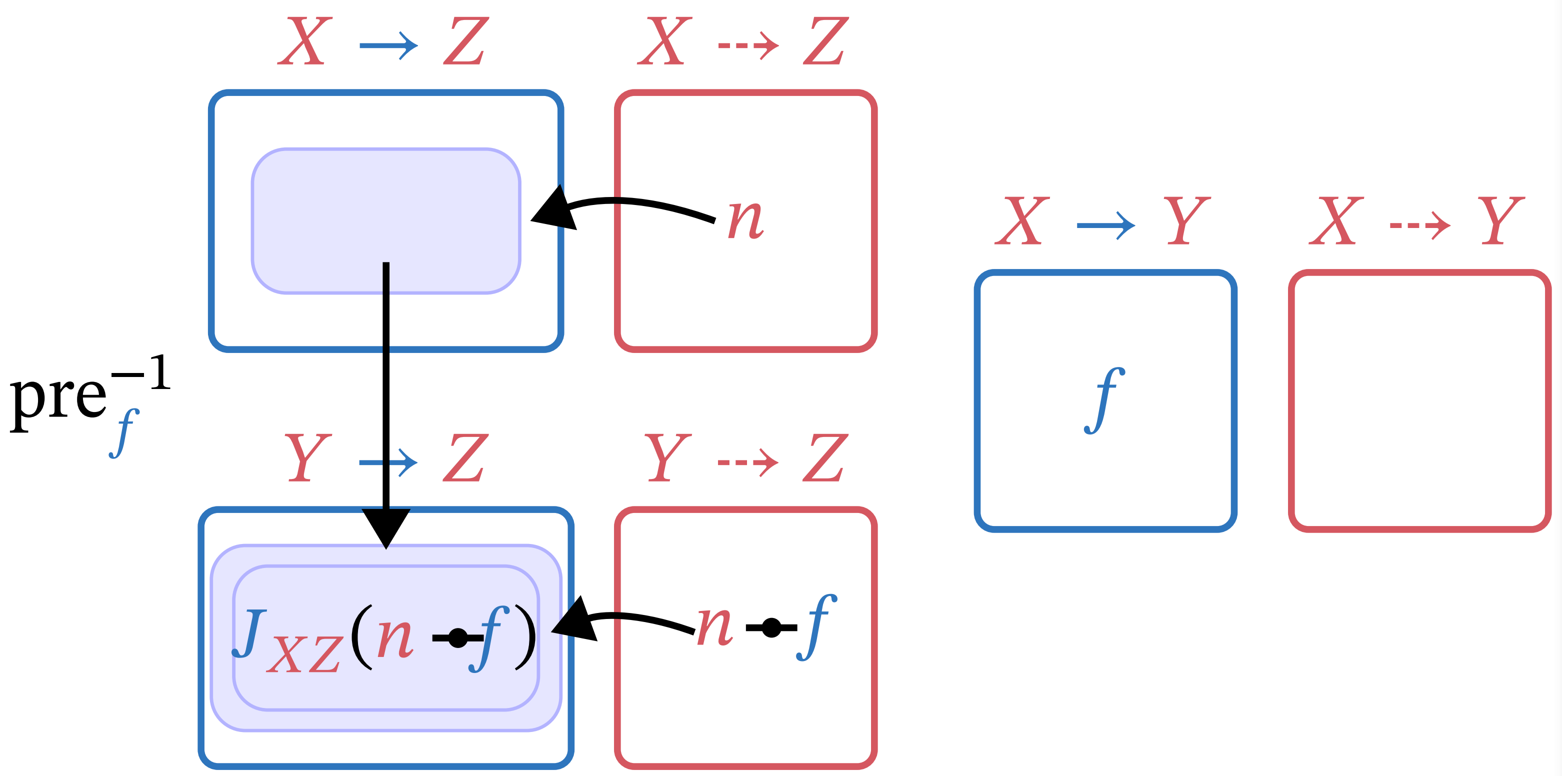
$$n \mathbin{-\!\!\bullet\!\!-} g = \emptyset$$

$$\begin{aligned} J_{XY} : \mathrm{Nom}_{\mathbf{C}}(X;Y) &\to \mathsf{Pow}(\mathrm{Hom}_{\mathbf{C}}(X;Y)) \\ n &\mapsto \{f \in \mathrm{Hom}_{\mathbf{C}}(X;Y) : i_{XY}(n, f)\} \end{aligned}$$









$$\frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \multimap n}{\dashrightarrow} Z}$$

Example: hiking on the Swiss mountains

Definition 5 (Berg). Let $h\colon \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ be a C^1 function, describing the elevation of a mountain. The set with elements $\langle a, b, h(a,b)\rangle$ is a manifold $\mathbb{M}$ that is embedded in $\mathbb{R}^3$. Let $\boldsymbol{\sigma} = [\sigma_{\mathrm{L}}, \sigma_{\mathrm{U}}] \subset \mathbb{R}$ be a closed interval of real numbers. The category $\mathbf{Berg}_{h,\sigma}$ is specified as follows:

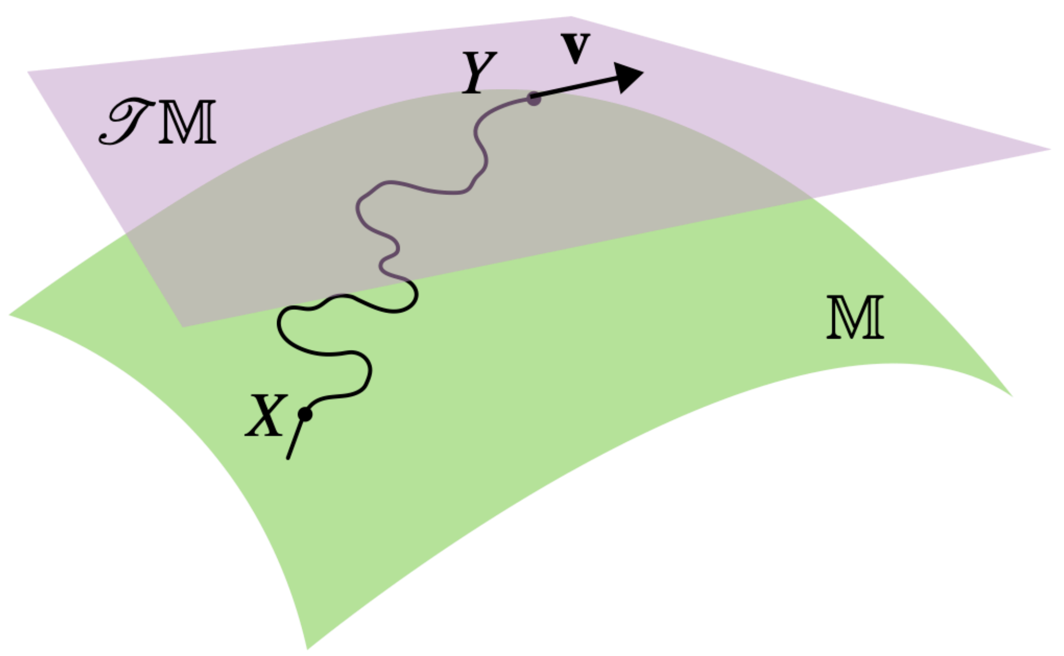
1. An object X is a pair $\langle \mathbf{p}, \mathbf{v}\rangle \in \mathscr{T}\mathbb{M}$, where $\mathbf{p} = \langle \mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z\rangle$ is the position, $\mathbf{v}$ is the velocity, and $\mathscr{T}\mathbb{M}$ is the tangent bundle of the manifold.
2. Morphisms are C^1 paths on the manifold.At each point of a path we define the *steepness* as:

$$s(\langle \mathbf{p}, \mathbf{v}\rangle) := \mathbf{v}_z / \sqrt{\mathbf{v}_x^2 + \mathbf{v}_y^2}. \tag{18}$$

   We choose as morphisms only the paths that have the steepness values contained in the interval $\boldsymbol{\sigma}$:

$$\mathrm{Hom}_{\mathbf{Berg}_{h,\sigma}}(X;Y) = \{f \text{ is a } C^1 \text{ path from } X \text{ to } Y \text{ and } s(f) \subseteq \boldsymbol{\sigma}\}, \tag{19}$$

3. Morphism composition is given by concatenation of paths.
4. Given any object, the identity morphism is the trivial self path with only one point.



Norphisms in Berg

- We take **norphisms** in **Berg** to be **lower bounds on the path distance:**

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \;:=\; \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

- A morphism is incompatible if it violates the lower bound:

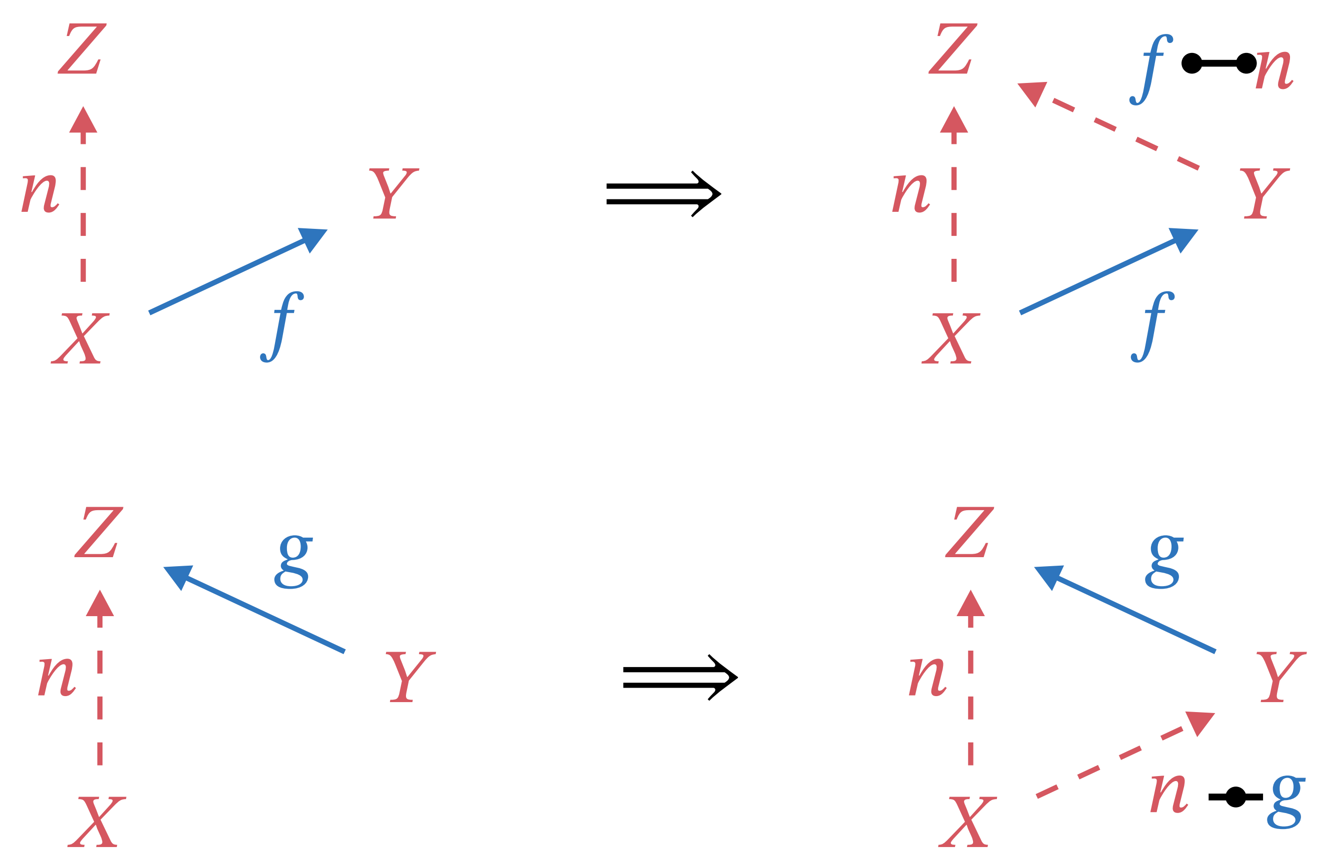
$$i_{XY}(n, f) = \mathrm{length}(f) < n$$

- An **optimal path** is a **feasible path** together with a **lower bound** on the distance:

$$\dfrac{f : X \to Y \quad \mathrm{length}(f) : X \dashrightarrow Y}{f \text{ is optimal}}$$

- **Norphism composition** rules:

$$f \bullet\!\!-\!\!\bullet\, n = \max\{n - \mathrm{length}(f), 0\}$$
$$n \,-\!\!\bullet\!-\, g = \max\{n - \mathrm{length}(g), 0\}$$



Norphism schemas for Berg

- The length of a path is never less than zero:

$$0 : X \dashrightarrow X$$

- The length of a path cannot be lower than the distance in 3D:

$$\|\mathbf{p}^1 - \mathbf{p}^2\| : \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle$$

- The length of a path cannot be lower than than the geodesic distance:

$$d_{\mathbb{M}}(\mathbf{p}^1, \mathbf{p}^2) : \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle$$

- The following bounds hold due to the constraint on inclination:

$$\frac{\mathbf{p}^1_z - \mathbf{p}^2_z < 0}{|\mathbf{p}^1_z - \mathbf{p}^2_z|/\sigma_{\mathrm{U}} : \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle} \qquad \frac{\mathbf{p}^1_z - \mathbf{p}^2_z > 0}{|\mathbf{p}^1_z - \mathbf{p}^2_z|/\sigma_{\mathrm{L}} : \langle \mathbf{p}^1, \mathbf{v}^1 \rangle \dashrightarrow \langle \mathbf{p}^2, \mathbf{v}^2 \rangle}$$

- Follow up: we can order schema axioms in partial order (“subnategories”?)

Different choices for norphisms for Berg

- We need to check the condition:

$$i_{YZ}(f \mathbin{\bullet\!\!-\!\!\bullet} n, g) \Rightarrow i_{XZ}(n, f \fatsemi g)$$
$$i_{XY}(n \mathbin{-\!\!\bullet\!\!-} g, f) \Rightarrow i_{XZ}(n, f \fatsemi g)$$

optionally, the exactness condition:

$$i_{YZ}(f \mathbin{\bullet\!\!-\!\!\bullet} n, g) \Leftrightarrow i_{XZ}(n, f \fatsemi g)$$
$$i_{XY}(n \mathbin{-\!\!\bullet\!\!-} g, f) \Leftrightarrow i_{XZ}(n, f \fatsemi g)$$

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \ := \ \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} n = \max\{n - \mathrm{length}(f), 0\}$$
$$n \mathbin{-\!\!\bullet\!\!-} g = \max\{n - \mathrm{length}(g), 0\}$$

✓ valid nategory

✗ not exact

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \ := \ \mathbb{R} \cup \{+\infty\}$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} n = n - \mathrm{length}(f)$$
$$n \mathbin{-\!\!\bullet\!\!-} g = n - \mathrm{length}(g)$$

✓ valid nategory

✓ exact

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \ := \ \mathbb{Z} \cup \{+\infty\}$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} n = \mathrm{floor}(n - \mathrm{length}(f))$$
$$n \mathbin{-\!\!\bullet\!\!-} g = \mathrm{floor}(n - \mathrm{length}(g))$$

✓ valid nategory

✗ not exact

$$\mathrm{Nom}_{\mathbf{Berg}}(X;Y) \ := \ \mathbb{Z} \cup \{+\infty\}$$

$$f \mathbin{\bullet\!\!-\!\!\bullet} n = \mathrm{round}(n - \mathrm{length}(f))$$
$$n \mathbin{-\!\!\bullet\!\!-} g = \mathrm{round}(n - \mathrm{length}(g))$$

✗ not a nategory

✗ not exact

Definition (Enriched category)

Let $\langle \mathbf{V}, \otimes, \mathbf{1}, as, lu, ru \rangle$ be a monoidal category, where *as* is the associator, *lu* is the left unitor, and *ru* is the right unitor.

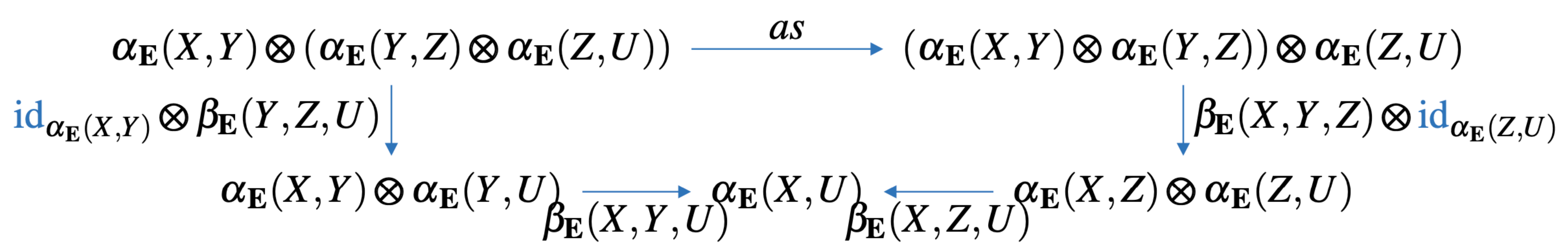
A $\mathbf{V}$-enriched category $\mathbf{E}$ is given by a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$, where

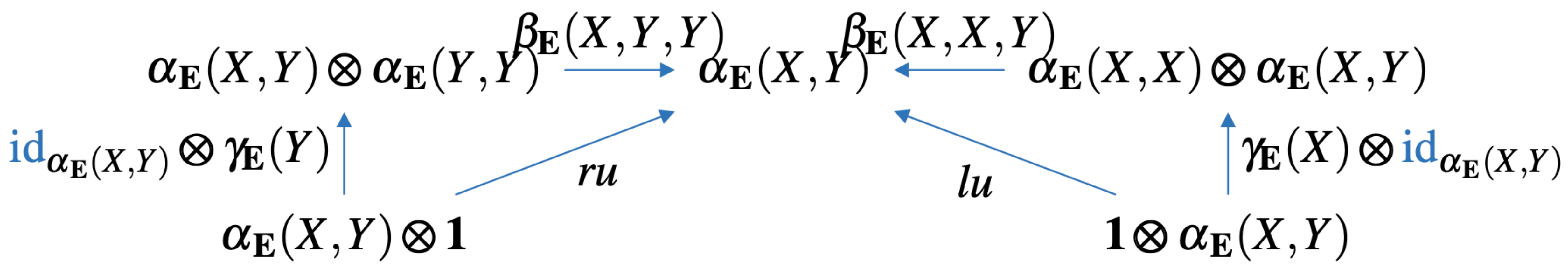
1. $\mathrm{Ob}_{\mathbf{E}}$ is a set of "objects".
2. $\alpha_{\mathbf{E}}$ is a function such that, for all pairs of objects $X, Y \in \mathrm{Ob}_{\mathbf{E}}$, the value $\alpha_{\mathbf{E}}(X,Y)$ is an object of $\mathbf{V}$.
3. $\beta_{\mathbf{E}}$ is a function such that, for all $X, Y, Z \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism $\beta_{\mathbf{E}}(X,Y,Z)$ of $\mathbf{V}$, called *composition morphism*:

$$\beta_{\mathbf{E}}(X,Y,Z) : \alpha_{\mathbf{E}}(X,Y) \otimes \alpha_{\mathbf{E}}(Y,Z) \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,Z).$$

4. $\gamma_{\mathbf{E}}$ is a function such that, for each $X \in \mathrm{Ob}_{\mathbf{E}}$, there exists a morphism of $\mathbf{V}$:

$$\gamma_{\mathbf{E}}(X) : \mathbf{1} \to_{\mathbf{V}} \alpha_{\mathbf{E}}(X,X).$$





Lemma. A category enriched in $\mathbf{P}$ gives the data necessary to define a small category, and vice versa.

Proof. We show one direction. Suppose that we are given a $\mathbf{P}$-enriched category as a tuple $\langle \mathrm{Ob}_{\mathbf{E}}, \alpha_{\mathbf{E}}, \beta_{\mathbf{E}}, \gamma_{\mathbf{E}} \rangle$. We can define a small category $\mathbf{C}$ as follows:

- Set $\mathrm{Ob}_{\mathbf{C}} := \mathrm{Ob}_{\mathbf{E}}$.
- For each $X, Y \in \mathrm{Ob}_{\mathbf{C}}$, let $\mathrm{Hom}_{\mathbf{C}}(X; Y) := \alpha_{\mathbf{E}}(X, Y)$.
- For each $X, Y, Z \in \mathrm{Ob}_{\mathbf{C}}$, we know a function

$$\beta_{\mathbf{E}}(X, Y, Z) : \mathrm{Hom}_{\mathbf{C}}(X; Y) \otimes \mathrm{Hom}_{\mathbf{C}}(Y; Z) \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X; Z). \tag{83}$$

  The diagrams constraints imply that this function is associative.
  Therefore, we use it to define morphism composition in $\mathbf{C}$, setting $\fatsemi_{X,Y,Z} := \beta_{\mathbf{E}}(X, Y, Z)$.
- For each $X \in \mathrm{Ob}_{\mathbf{C}}$ we know a function $\gamma_{\mathbf{E}}(X) : 1 \to_{\mathbf{Set}} \mathrm{Hom}_{\mathbf{C}}(X; X)$ that selects a morphism. The diagrams constraints imply that such morphism satisfies unitality with respect to $\fatsemi_{X,Y,Z}$. Therefore, we can use it to define the identity at each object:

$$\mathrm{id}_X := \gamma_{\mathbf{E}}(X)(\bullet). \tag{84}$$

□

The G(C) construction

- The **G(C)** construction is due to De Paiva.
- It provides a nontrivial model of **linear logic:** all 4 connectives, 4 units, negations, and modalities are distinct.
- See the recent post by Niu on the Topos website that clarifies the relation between **G(C)** and **Poly.**

- Plan:
  - We recall the definition of **G(Set);**
  - We recall some of the monoidal products defined by De Paiva;
  - We will define *yet another one*;
  - We will use it as a target for enrichment.

Definition (**PN**)
We call **PN** the monoidal category $\langle \mathbf{G}(\mathbf{Set}, \mathbf{Bool}), \sqcup \rangle$.

Definition (**G**(**Set**))
An object of **G**(**Set**) is a tuple

$$\langle Q, A, C \rangle,$$

where: Q is a set, A is a set, $C : Q \rightarrow_{\mathbf{Rel}} A$ is a relation.
A morphism $\mathbf{r} : \langle Q_1, A_1, C_1 \rangle \rightarrow_{\mathbf{GC}} \langle Q_2, A_2, C_2 \rangle$ is a pair of maps

$$\begin{aligned} \mathbf{r} &= \langle r_\flat, r^\sharp \rangle, \\ r_\flat &: Q_1 \leftarrow_{\mathbf{Set}} Q_2, \\ r^\sharp &: A_1 \rightarrow_{\mathbf{Set}} A_2, \end{aligned}$$

that satisfy the property

$$\forall q_2 : Q_2 \quad \forall a_1 : A_1 \quad r_\flat(q_2) C_1 \, a_1 \;\Rightarrow\; q_2 C_2 \, r^\sharp(a_1).$$

Morphism composition is defined component-wise:

$$\begin{aligned} (\mathbf{r} \mathbin{⨾} \mathbf{s})_\flat &= s_\flat \mathbin{⨾} r_\flat, \\ (\mathbf{r} \mathbin{⨾} \mathbf{s})^\sharp &= r^\sharp \mathbin{⨾} s^\sharp. \end{aligned}$$

The identity at $\langle Q, A, C \rangle$ is given by $\mathrm{id}_{\langle Q, A, C \rangle} = \langle \mathrm{id}_Q, \mathrm{id}_A \rangle$.

Definition (Category **G**(**Set**, **B**))
Let **B** be a category with finite products and coproducts. An object of the category **G**(**Set**, **B**) is a tuple

$$\langle Q, A, \kappa \rangle,$$

where Q is a set; A is a set, κ is a function

$$\kappa : Q \times A \to \mathrm{Ob}_{\mathbf{B}}.$$

A morphism $\mathbf{r} : \langle Q_1, A_1, \kappa_1 \rangle \to \langle Q_2, A_2, \kappa_2 \rangle$ is a tuple of three functions

$$\begin{aligned} \mathbf{r} = & \langle r_\flat, r^\sharp, r^* \rangle, \\ & r_\flat : Q_1 \leftarrow_{\mathbf{Set}} Q_2, \\ & r^\sharp : A_1 \to_{\mathbf{Set}} A_2, \\ & r^* : \{q_2 : Q_2, a_1 : A_1\} \to \kappa_1(r_\flat(q_2), a_1) \to_{\mathbf{B}} \kappa_2(q_2, r^\sharp(a_1)). \end{aligned}$$

The composition of the above morphism $\mathbf{r}$ with $\mathbf{s} : \langle Q_2, A_2, \kappa_2 \rangle \to \langle Q_3, A_3, \kappa_3 \rangle$ is defined as follows:

$$\begin{aligned} (\mathbf{r} \fatsemi \mathbf{s})_\flat &= s_\flat \fatsemi r_\flat, \\ (\mathbf{r} \fatsemi \mathbf{s})^\sharp &= r^\sharp \fatsemi s^\sharp, \\ (\mathbf{r} \fatsemi \mathbf{s})^* &: \langle q_3, a_1 \rangle \mapsto r^*(s_\flat(q_3), a_1) \fatsemi_{\mathbf{B}} s^*(q_3, r^\sharp(a_1)). \end{aligned}$$

More explicitly,

$$(\mathbf{r} \fatsemi \mathbf{s})^* : \langle q_3, a_1 \rangle \mapsto$$

$$\kappa_1((s_\flat \fatsemi r_\flat)(q_3), a_1) \xrightarrow{r^*(s_\flat(q_3), a_1)} \kappa_2(s_\flat(q_3), r^\sharp(a_1)) \xrightarrow{s^*(q_3, r^\sharp(a_1))} \kappa_3(q_3, (r^\sharp \fatsemi s^\sharp)(a_1)).$$

The identity at $\langle Q, A, \kappa \rangle$ is given by $\langle \mathrm{id}_Q, \mathrm{id}_A, \langle q, a \rangle \mapsto \mathrm{id}_{\kappa(q,a)} \rangle$.

Definition ($\mathbf{G}(\mathbf{Cat}, \mathbf{B})$)
Given a category $\mathbf{B}$, an object of $\mathbf{G}(\mathbf{Cat}, \mathbf{B})$ is a tuple

$$\langle Q, A, \varkappa \rangle,$$

where Q is a category, A is a category, $\varkappa$ is a functor

$$\varkappa : Q^{\mathrm{op}} \times A \to \mathbf{B}.$$

A morphism $\mathbf{r} : \langle Q_1, A_1, \varkappa_1 \rangle \to_{\mathbf{GCat}} \langle Q_2, A_2, \varkappa_2 \rangle$ is a tuple

$$\mathbf{r} = \langle r_\flat, r^\sharp, r^* \rangle,$$

where

- $r_\flat : Q_2 \to_{\mathbf{Cat}} Q_1$ is a functor,
- $r^\sharp : A_1 \to_{\mathbf{Cat}} A_2$ is a functor,
- r^* is a natural transformation between two functors

$$F, G : Q_2^{\mathrm{op}} \times A_1 \to \mathbf{B},$$

defined as

$$F = (r_\flat \times \mathrm{id}_{A_1}) \mathbin{⨾} \varkappa_1,$$
$$G = (\mathrm{id}_{Q_2^{\mathrm{op}}} \times r^\sharp) \mathbin{⨾} \varkappa_2.$$

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Given a category $\mathbf{B}$, an object of $\mathbf{G}(\mathbf{Cat}, \mathbf{B})$ is a tuple

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A morphism $\mathbf{r} : \langle Q_1, A_1, \varkappa_1 \rangle \to_{\mathbf{GCat}} \langle Q_2, A_2, \varkappa_2 \rangle$ is a tuple

$$\mathbf{r} = \langle r_\flat, r^\sharp, r^* \rangle,$$

where

not weird anymore!

- $r_\flat : Q_2 \to_{\mathbf{Cat}} Q_1$ is a functor,
- $r^\sharp : A_1 \to_{\mathbf{Cat}} A_2$ is a functor,
- r^* is a natural transformation between two functors

$$F, G : Q_2^{\mathrm{op}} \times A_1 \to \mathbf{B},$$

  defined as

$$F = (r_\flat \times \mathrm{id}_{A_1}) ⨟ \varkappa_1,$$
$$G = (\mathrm{id}_{Q_2^{\mathrm{op}}} \times r^\sharp) ⨟ \varkappa_2.$$

A monoidal product

Definition (Monoidal product $*$)
The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1\rangle *^{\cdot} \langle Q_2, A_2, \kappa_2\rangle = \langle Q_1 \times Q_2, A_1 \times A_2, \kappa_1 * \kappa_2\rangle$$

$$\kappa_1 * \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle \mapsto \kappa_1(q_1, a_1) \times^{\cdot}_{\mathbf{B}} \kappa_2(q_2, a_2),$$

where $\times^{\cdot}_{\mathbf{B}}$ is the product of two objects in $\mathbf{B}$. The monoidal unit is

$$1_* = \langle\{\bullet\}, \{\bullet\}, \top\rangle, \qquad \top : \langle\bullet, \bullet\rangle \mapsto 1_{\mathbf{B}}.$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$ is

$$\mathbf{r} *^{\to} \mathbf{s} : \langle Q_1 \times Q_2, A_1 \times A_2, \kappa_1 * \kappa_2\rangle \to \langle Q_3 \times Q_4, A_3 \times A_4, \kappa_3 * \kappa_4\rangle$$

$$(\mathbf{r} *^{\to} \mathbf{s})_\flat = r_\flat \times^{\to} s_\flat,$$

$$(\mathbf{r} *^{\to} \mathbf{s})^\sharp = r^\sharp \times^{\to} s^\sharp,$$

$$(\mathbf{r} *^{\to} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*(q_3, a_1) \times^{\to}_{\mathbf{B}} s^*(q_4, a_2).$$

... another one...

Definition (Monoidal product $\otimes$)
The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1\rangle \otimes^{\cdot} \langle Q_2, A_2, \kappa_2\rangle = \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \otimes \kappa_2\rangle$$

$$\kappa_1 \otimes \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle \mapsto \kappa_1(q_1(a_2), a_1) \times^{\cdot}_{\mathbf{B}} \kappa_2(q_2(a_1), a_2),$$

where $\times^{\cdot}_{\mathbf{B}}$ is the product of two objects in **B**. The monoidal unit is

$$1_{\otimes} = \langle\{\bullet\}, \{\bullet\}, \top\rangle, \qquad \top : \langle\bullet, \bullet\rangle \mapsto 1_{\mathbf{B}}.$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$ is

$$\mathbf{r} \otimes^{\rightarrow} \mathbf{s} : \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \otimes \kappa_2\rangle \to \langle Q_3^{A_4} \times Q_4^{A_3}, A_3 \times A_4, \kappa_3 \otimes \kappa_4\rangle$$

$$(\mathbf{r} \otimes^{\rightarrow} \mathbf{s})_\flat = \langle s^\sharp \mathbin{⨾} - \mathbin{⨾} r_\flat, r^\sharp \mathbin{⨾} - \mathbin{⨾} s_\flat\rangle,$$

$$(\mathbf{r} \otimes^{\rightarrow} \mathbf{s})^\sharp = r^\sharp \times^{\rightarrow} s^\sharp,$$

$$(\mathbf{r} \otimes^{\rightarrow} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*((s^\sharp \mathbin{⨾} q_3)(a_2), a_1) \times^{\rightarrow}_{\mathbf{B}} s^*((r^\sharp \mathbin{⨾} q_3)(a_1), a_2).$$

... and another one...

Definition (Monoidal product $\parr$)
The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1 \rangle \parr^{\cdot} \langle Q_2, A_2, \kappa_2 \rangle = \langle Q_1 \times Q_2, A_1^{Q_2} \times A_2^{Q_1}, \kappa_1 \parr \kappa_2 \rangle$$

$$\kappa_1 \parr \kappa_2 : \langle \langle q_1, q_2 \rangle, \langle a_1, a_2 \rangle \rangle \mapsto \kappa_1(q_1(a_2), a_1) +^{\cdot}_{\mathbf{B}} \kappa_2(q_2(a_1), a_2),$$

where $+^{\cdot}_{\mathbf{B}}$ is the coproduct of two objects in $\mathbf{B}$. The monoidal unit is

$$1_{\parr} = \langle \{\bullet\}, \{\bullet\}, \bot \rangle, \qquad \bot : \langle \bullet, \bullet \rangle \mapsto 0_{\mathbf{B}}.$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1 \rangle \to \langle Q_3, A_3, \kappa_3 \rangle$, $\mathbf{s} : \langle Q_2, A_2, \kappa_2 \rangle \to \langle Q_4, A_4, \kappa_4 \rangle$ is

$$\mathbf{r} \parr^{\to} \mathbf{s} : \langle Q_1 \times Q_2, A_1^{Q_2} \times A_2^{Q_1}, \kappa_1 \parr \kappa_2 \rangle \to \langle Q_3 \times Q_4, A_3^{Q_4} \times A_4^{Q_3}, \kappa_3 \parr \kappa_4 \rangle$$

$$(\mathbf{r} \parr^{\to} \mathbf{s})_\flat = r_\flat \times^{\to} s_\flat,$$

$$(\mathbf{r} \parr^{\to} \mathbf{s})^\sharp = \langle s_\flat \,⨟ - ⨟\, r^\sharp, r_\flat \,⨟ - ⨟\, s^\sharp \rangle,$$

$$(\mathbf{r} \parr^{\to} \mathbf{s})^* : \langle \langle q_3, q_4 \rangle, \langle a_1, a_2 \rangle \rangle \mapsto r^*(\dots, a_1) +^{\to}_{\mathbf{B}} s^*(\dots, a_2).$$

where $+^{\to}_{\mathbf{B}}$ is the coproduct of two morphisms in $\mathbf{B}$.

...and the one we need!

Definition (Monoidal product $\sqcup$)
The action on the objects is defined as follows:

$$\langle Q_1, A_1, \kappa_1\rangle \mathbin{\dot{\sqcup}} \langle Q_2, A_2, \kappa_2\rangle = \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \sqcup \kappa_2\rangle$$

$$\kappa_1 \sqcup \kappa_2 : \langle\langle q_1, q_2\rangle, \langle a_1, a_2\rangle\rangle \mapsto \kappa_1(q_1(a_2), a_1) \mathbin{\dot{+}_{\mathbf{B}}} \kappa_2(q_2(a_1), a_2)$$

The monoidal unit is

$$1_{\sqcup} = \langle\{\bullet\}, \{\bullet\}, \bot\rangle, \qquad \bot : \langle \bullet, \bullet\rangle \mapsto 0_{\mathbf{B}}.$$

The product of $\mathbf{r} : \langle Q_1, A_1, \kappa_1\rangle \to \langle Q_3, A_3, \kappa_3\rangle$ and $\mathbf{s} : \langle Q_2, A_2, \kappa_2\rangle \to \langle Q_4, A_4, \kappa_4\rangle$ is

$$\mathbf{r} \mathbin{\vec{\sqcup}} \mathbf{s} : \langle Q_1^{A_2} \times Q_2^{A_1}, A_1 \times A_2, \kappa_1 \sqcup \kappa_2\rangle \to \langle Q_3^{A_4} \times Q_4^{A_3}, A_3 \times A_4, \kappa_3 \sqcup \kappa_4\rangle$$

$$(\mathbf{r} \mathbin{\vec{\sqcup}} \mathbf{s})_\flat = \langle s^\sharp \mathbin{⨟} - \mathbin{⨟} r_\flat, r^\sharp \mathbin{⨟} - \mathbin{⨟} s_\flat\rangle,$$

$$(\mathbf{r} \mathbin{\vec{\sqcup}} \mathbf{s})^\sharp = r^\sharp \mathbin{\vec{\times}} s^\sharp,$$

$$(\mathbf{r} \mathbin{\vec{\sqcup}} \mathbf{s})^* : \langle\langle q_3, q_4\rangle, \langle a_1, a_2\rangle\rangle \mapsto r^*(s^\sharp \mathbin{⨟} q_3(a_2), a_1) \mathbin{\vec{+}_{\mathbf{B}}} s^*(r^\sharp \mathbin{⨟} q_3(a_1), a_2).$$

Norphisms by enrichment

Definition (**PN**)
We call **PN** the monoidal category $\langle \mathbf{G}(\mathbf{Set}, \mathbf{Bool}), \sqcup \rangle$.

Proposition. A **PN**-enriched category provides the data necessary to specify a nategory. However, not all nategories can be specified by the data of a **PN**-enriched category, because the nategory produced has two additional neutrality properties:

$$\mathrm{id}_X \mathbin{\bullet\!\!-\!\!\bullet} n = n, \tag{neut-1}$$

$$n \mathbin{-\!\!\bullet\!\!-} \mathrm{id}_Y = n, \tag{neut-2}$$

two "distributivity" conditions:

$$(f \mathbin{⨟} g) \mathbin{\bullet\!\!-\!\!\bullet} n = g \mathbin{\bullet\!\!-\!\!\bullet} (f \mathbin{\bullet\!\!-\!\!\bullet} n), \tag{dist-1}$$

$$n \mathbin{-\!\!\bullet\!\!-} (g \mathbin{⨟} h) = (n \mathbin{-\!\!\bullet\!\!-} h) \mathbin{-\!\!\bullet\!\!-} g, \tag{dist-2}$$

and a "mixed associativity" condition

$$f \mathbin{\bullet\!\!-\!\!\bullet} (n \mathbin{-\!\!\bullet\!\!-} h) = (f \mathbin{\bullet\!\!-\!\!\bullet} n) \mathbin{-\!\!\bullet\!\!-} h, \tag{assoc}$$

which are not necessarily satisfied by all nategories.

Some steps from the proof

- The enrichment gives, **for each pair of objects *X*, *Y***, a tuple

$$\alpha_{\mathbf{E}}(X,Y) = \langle Q, A, \kappa \rangle$$

which we use to define Hom, Nom, and i:

$$\alpha_{\mathbf{E}}(X,Y) = \langle \mathrm{Nom}_{\mathbf{C}}(X;Y), \mathrm{Hom}_{\mathbf{C}}(X;Y), i_{XY} \rangle$$

- **For each object *X***, we have a morphism

$$\gamma_{\mathbf{E}}(X) : \mathbf{1}_{\mathbf{PN}} \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X,X)$$

in our case:

$$\mathbf{r} = \gamma_{\mathbf{E}}(X) : \langle \{\bullet\}, \{\bullet\}, \bot \rangle \to_{\mathbf{PN}} \langle \mathrm{Hom}_{\mathbf{C}}(X;X), \mathrm{Nom}_{\mathbf{C}}(X;X), i_{XX} \rangle$$

the forward part picks a morphism that, given the other conditions, is the identity. The other conditions are vacuous.

- Next up: composition operations..

Derivation of morphism composition operations

- **For each triple *X, Y, Z*,** enrichment gives a morphism of PN

$$\beta_{\mathbf{E}}(X,Y,Z) : \alpha_{\mathbf{E}}(X,Y) \otimes_{\mathbf{PN}} \alpha_{\mathbf{E}}(Y,Z) \to_{\mathbf{PN}} \alpha_{\mathbf{E}}(X,Z)$$

unrolling:

$$\mathbf{s}_{XYZ} : \langle \mathrm{N}_{XY}, \mathrm{H}_{XY}, i_{XY} \rangle \otimes_{\mathbf{PN}} \langle \mathrm{N}_{YZ}, \mathrm{H}_{YZ}, i_{YZ} \rangle \to_{\mathbf{PN}} \langle \mathrm{N}_{XZ}, \mathrm{H}_{XZ}, i_{XZ} \rangle$$

$$\mathbf{s}_{XYZ} : \langle {\mathrm{N}_{XY}}^{\mathrm{H}_{YZ}} \times {\mathrm{N}_{YZ}}^{\mathrm{H}_{XY}}, \mathrm{H}_{XY} \times \mathrm{H}_{YZ}, i_{XY} \sqcup i_{YZ} \rangle \to_{\mathbf{PN}} \langle \mathrm{N}_{XZ}, \mathrm{H}_{XZ}, i_{XZ} \rangle$$

- The forward part recovers morphism composition:

$$s^{\sharp} : \mathrm{Hom}_{\mathbf{C}}(X;Y) \times \mathrm{Hom}_{\mathbf{C}}(Y;Z) \to \mathrm{Hom}_{\mathbf{C}}(X;Z)$$

- The backward part gives the morphism composition functions:

$$s_{\flat} : \mathrm{N}_{XZ} \to {\mathrm{N}_{XY}}^{\mathrm{H}_{YZ}} \times {\mathrm{N}_{YZ}}^{\mathrm{H}_{XY}} \quad \longrightarrow \quad \begin{array}{l} -\!\bullet : \mathrm{N}_{XZ} \times \mathrm{H}_{YZ} \to \mathrm{N}_{XY}, \\ \bullet\!\!-\!\!\bullet : \mathrm{H}_{XY} \times \mathrm{N}_{XZ} \to \mathrm{N}_{YZ} \end{array}$$

- The last component can be evaluated to get:

$$s^{*}(n, \langle f, g \rangle) : (i_{XY} \sqcup i_{YZ})(\langle (n \mathbin{-\!\bullet} -), (- \mathbin{\bullet\!\!-\!\!\bullet} n) \rangle, \langle f, g \rangle) \to_{\mathbf{Bool}} i_{XZ}(n, f \mathbin{⨟} g)$$

expanding:

$$s^{*}(n, \langle f, g \rangle) : i_{XY}(n \mathbin{-\!\bullet} g, f) \dot{+}_{\mathbf{Bool}} i_{YZ}(f \mathbin{\bullet\!\!-\!\!\bullet} n, g) \to_{\mathbf{Bool}} i_{XZ}(n, f \mathbin{⨟} g)$$

which is equivalent to 2 morphisms:

$$\begin{array}{l} s^{*}_{1}(n, \langle f, g \rangle) : i_{XY}(n \mathbin{-\!\bullet} g, f) \to_{\mathbf{Bool}} i_{XZ}(n, f \mathbin{⨟} g) \\ s^{*}_{2}(n, \langle f, g \rangle) : i_{YZ}(f \mathbin{\bullet\!\!-\!\!\bullet} n, g) \to_{\mathbf{Bool}} i_{XZ}(n, f \mathbin{⨟} g) \end{array} \quad \longrightarrow \quad \begin{array}{l} i_{YZ}(f \mathbin{\bullet\!\!-\!\!\bullet} n, g) \Rightarrow i_{XZ}(n, f \mathbin{⨟} g) \\ i_{XY}(n \mathbin{-\!\bullet} g, f) \Rightarrow i_{XZ}(n, f \mathbin{⨟} g) \end{array}$$

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two “distributivity” conditions:

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$$n \mathbin{-\!\bullet\!-} (g \mathbin{;} h) = (n \mathbin{-\!\bullet\!-} h) \mathbin{-\!\bullet\!-} g, \tag{dist-2}$$

and a “mixed associativity” condition

$$f \mathbin{\bullet\!\!-\!\!\bullet} (n \mathbin{-\!\bullet\!-} h) = (f \mathbin{\bullet\!\!-\!\!\bullet} n) \mathbin{-\!\bullet\!-} h, \tag{assoc}$$

which are not necessarily satisfied by all nategories.

DP

- A morphism in **DP** is a **design problem**, an **expert**

> **Definition** (Design Problem)
> A *design problem* (DP) is a tuple $\langle \mathbf{F}, \mathbf{R}, \mathbf{d} \rangle$, where $\mathbf{F}$, $\mathbf{R}$ are posets and $\mathbf{d}$ is a monotone map of the form
> $$\mathbf{d} : \mathbf{F}^{\mathrm{op}} \times \mathbf{R} \to_{\mathbf{Pos}} \mathbf{Bool}.$$

- The composition in **DP** is given by

> **Definition** (Series composition)
> Let $\mathbf{d} : \mathbf{P} \nrightarrow \mathbf{Q}$ and $\mathbf{e} : \mathbf{Q} \nrightarrow \mathbf{R}$ be design problems. We define their *series composition* $(\mathbf{d} \mathbin{;} \mathbf{e}) : \mathbf{P} \nrightarrow \mathbf{R}$ as:
> $$\begin{aligned} (\mathbf{d} \mathbin{;} \mathbf{e}) : \mathbf{P}^{\mathrm{op}} \times \mathbf{R} &\to_{\mathbf{Pos}} \mathbf{Bool} \\ \langle p^*, r \rangle &\mapsto \bigvee_{q \in \mathbf{Q}} \mathbf{d}(p^*, q) \wedge \mathbf{e}(q^*, r). \end{aligned}$$

- Norphisms (**n**esign problems) are **infeasibility** relations

Example: you cannot build a perpetual motion machine

- These are still monotone maps, now stating **infeasibility**

$$n : \mathbf{F} \times \mathbf{R}^{\mathrm{op}} \to_{\mathbf{Pos}} \mathbf{Bool}.$$

Morphisms and norphisms in DP

- Start from $d: \mathbf{F} \nrightarrow \mathbf{R}$ and $n: \mathbf{F} \nrightarrow \mathbf{R}$.

- **Compatibility** ensures that there are **no contradictions**

$$i_{\mathbf{FR}}(n, d) = \exists f \in \mathbf{F}, r \in \mathbf{R}: d(f, r) \wedge n(f, r)$$

- How do **design** problems and **nesign** problems compose?

- Starting from $n: \mathbf{P} \nrightarrow \mathbf{Q}$ and $d: \mathbf{R} \nrightarrow \mathbf{Q}$.

$$(n \mathbin{-\!\bullet} d)(p, r) = \bigvee_{q \in \mathbf{Q}} n(p, q) \wedge d(r, q).$$

- Starting from $d: \mathbf{Q} \nrightarrow \mathbf{P}$ and $n: \mathbf{Q} \nrightarrow \mathbf{R}$

$$(d \mathbin{\bullet\!-\!\bullet} n)(p, r) = \bigvee_{q \in \mathbf{Q}} d(q, p) \wedge n(q, r).$$

Morphisms and norphisms in DP

- Let's consider the example of two dams

- Consider posets $\mathbf{P} = \mathbf{Q} = \mathbf{R} = \langle \mathbb{R}_{[\mathrm{J}]}, \leq \rangle$

- Dams transform **potential energy** into **kinetic energy**

- Let's say we have feasibility and infeasibility information about a dam

$$d : \mathbf{R} \nrightarrow \mathbf{Q}, \qquad n : \mathbf{P} \nrightarrow \mathbf{Q}$$

$$\frac{d(r, q)}{r \cdot 1.1 \leq q}, \qquad \frac{n(p, q)}{p \cdot 1.2 > q}.$$

- These produce a **nesign** problem $(n \mathbin{-\!\bullet\!-} d) : \mathbf{P} \nrightarrow \mathbf{R}$ describing infeasibility between kinetic energies: can I get 10 J from 9 J? No!

$$\begin{aligned}(n \mathbin{-\!\bullet\!-} d)(10, 9) &= \bigvee_{q \in \mathbf{Q}} n(10, q) \wedge d(9, q) \\ &= \bigvee_{q \in \mathbf{Q}} (9.9 \leq q < 12) = \top.\end{aligned}$$

Conclusions and future work

- **Negative information can be categorified** using negative arrows (norphisms).
  - (as opposed to using some logic on top of category theory...)
- Norphisms behave fundamentally differently than morphisms.
  They compose using morphisms as catalysts.

$$\frac{f : X \to Y \quad g : Y \to Z}{(f \mathbin{;} g) : X \to Z} \qquad \frac{Y \xleftarrow{f} X \overset{n}{\dashrightarrow} Z}{Y \overset{f \bullet\!-\!\bullet\, n}{\dashrightarrow} Z} \qquad \frac{X \overset{n}{\dashrightarrow} Z \xleftarrow{g} Y}{X \overset{n\, -\!\bullet\!-\, g}{\dashrightarrow} Y}$$

- **“Nategories” generalize categories** to account for the norphism machinery.

- We can derive the norphism rules very elegantly using **enriched category theory.**
  - Just like a **Set**-enriched category provides the data for a small category, ...
  - ... a **PN**-enriched category provides the data for a nategory.

- **Future work**
  - **PN** enrichment is too strong; induces more properties.
  - Surveying natural norphism structures in the wild.
  - Explore more the idea of algorithms producing both positive and negative information.
  - Generalization to higher-level concepts. What would a “nunctor” be?