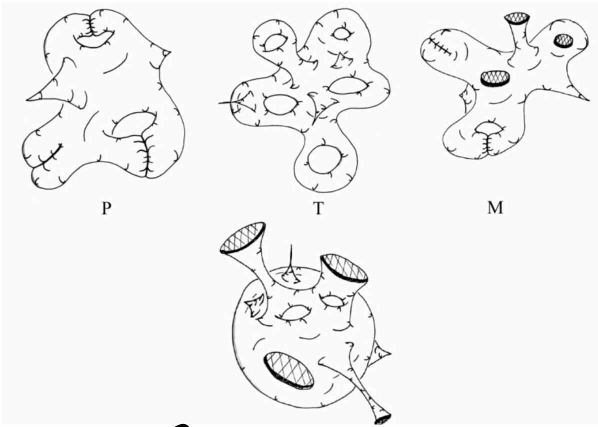
A Synthetic Approach



To Orbifolds

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The Plan

-2) What is an orbifold?

-1) Homotopy Type Theory

0) Groups in HoTT

1) Examples of Orbifolds

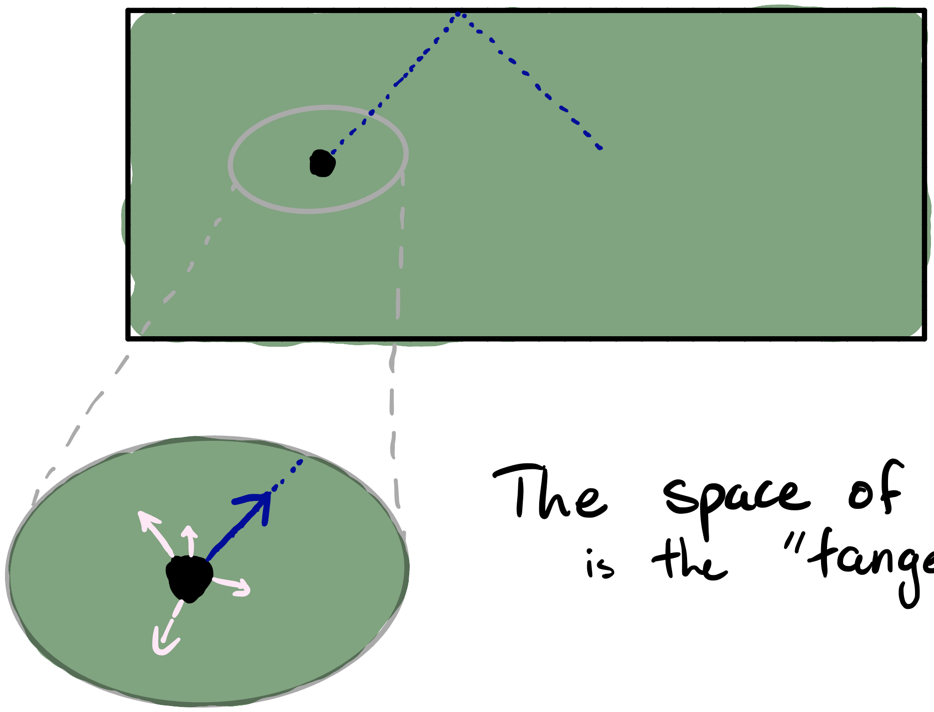
2) Homotopy types of Orbifolds

3) Synthetic differential geometry

arXiv: 2205.15887

A billiards table

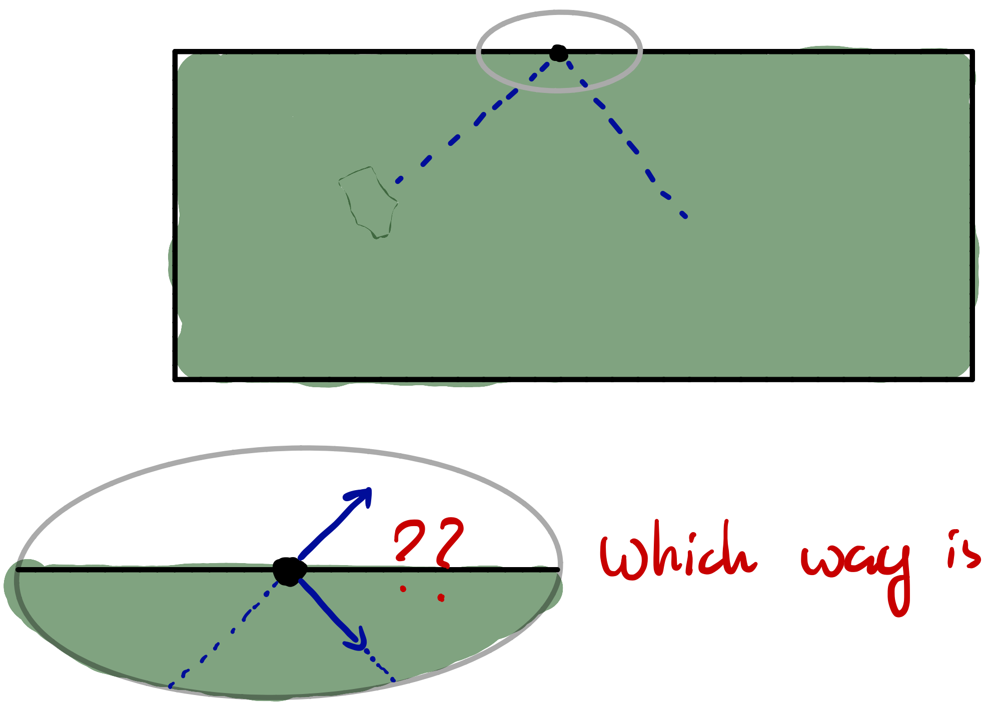
w/ point balls and elastic collisions



The space of directions is the "tangent space"

A billiards table

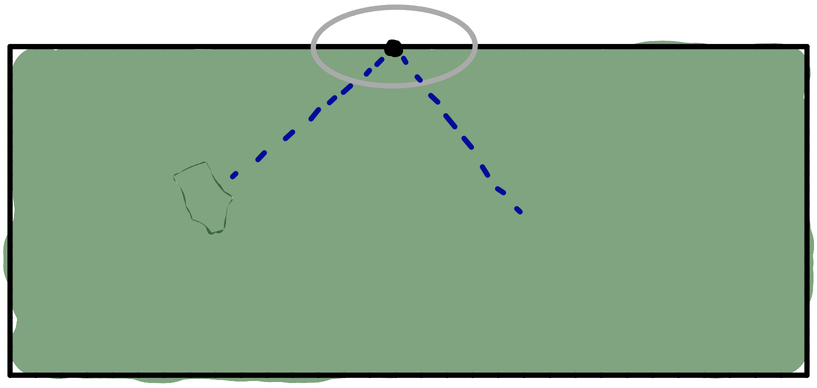
w/ point balls and elastic collisions

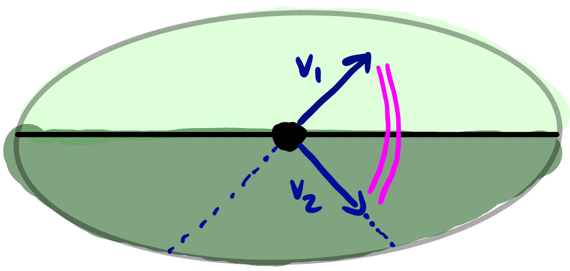


which way is it going?

A billiards table

w/ point balls and elastic collisions



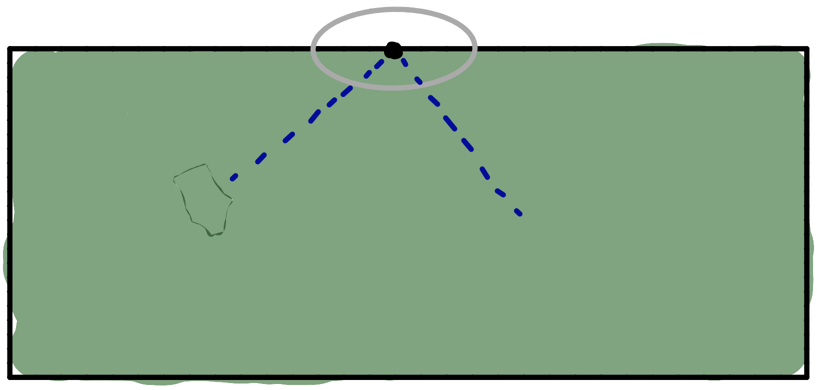


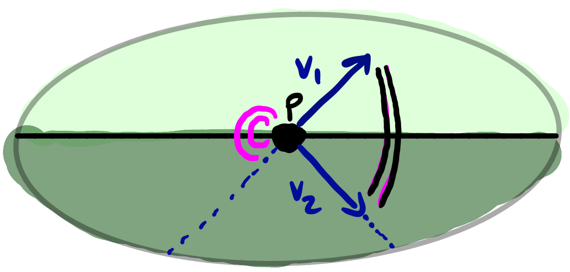
Both ways

bounce : $V_1 = V_2$

A billiards table

w/ point balls and elastic collisions

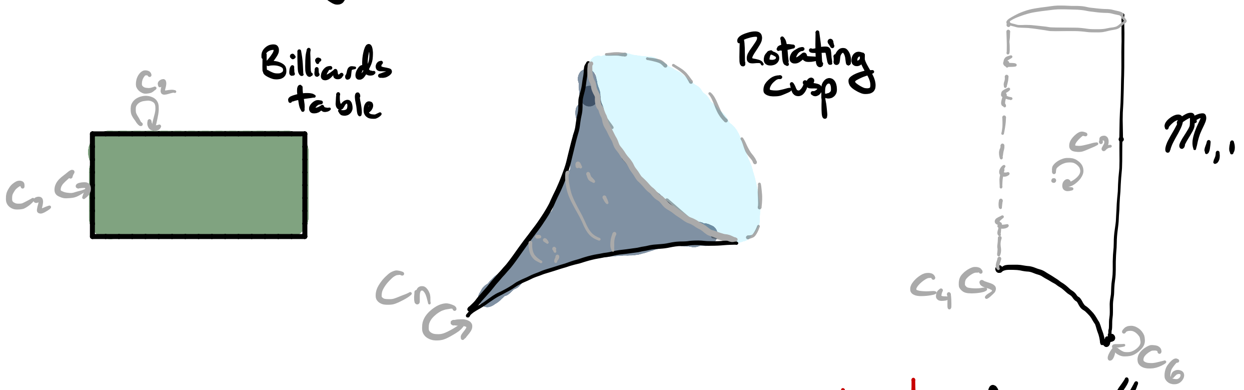




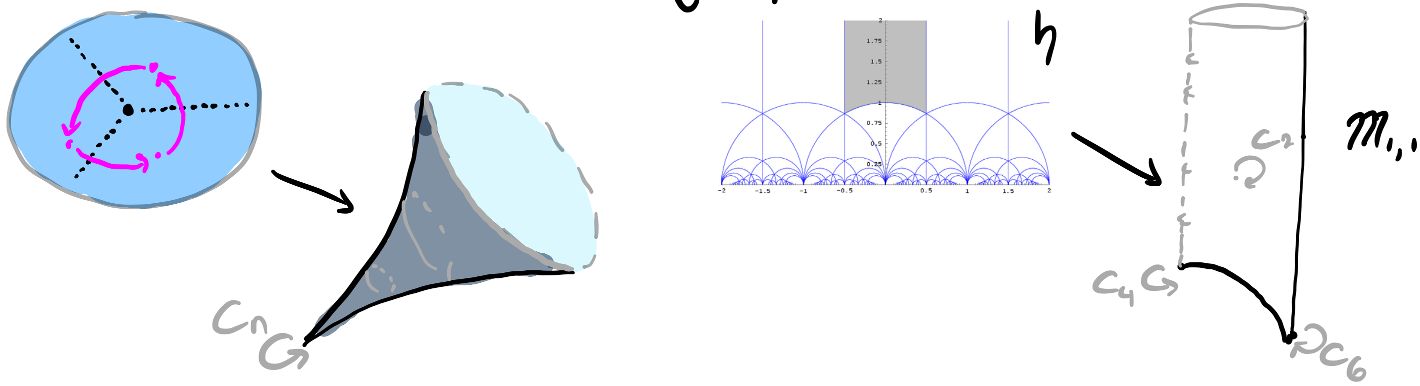
reflect : $p = p$

bounce : $V_1 = V_2$ over reflect

Orbifolds are "smooth spaces" where the points have finite symmetries.



A good orbifold is the homotopy quotient of a "smooth space" by the action of a discrete group.



[Orbifolds are "smooth spaces" where the points have finite symmetries.]

In set theoretical foundations, points can't have internal symmetries, so we have to carry around this data separately.

But in homotopy type theory, they can!

Eg: $\{0,1\} : \mathrm{Set}$, and

$(\{0,1\} =_{\mathrm{Set}} \{0,1\})$ is isomorphic to $\mathbb{Z}/2$

or $\mathbb{R}^n : \mathrm{VectorSpace}_{\mathbb{R}}$

$(\mathbb{R}^n =_{\mathrm{Vect}_{\mathbb{R}}} \mathbb{R}^n)$ is isomorphic to $GL_n(\mathbb{R})$

Orbifolds, classically:

"weak quotient"

If $\Gamma \circlearrowright X$ proper discontinuously, $X /\!/ \Gamma$ is a "good" orbifold.

$X /\!/ \Gamma$ is presented by the action groupoid

$$\begin{array}{ccc} (X/\!/\Gamma)_1 & := & \{(x,y,\gamma) \mid \gamma \cdot x = y\} \\ s \downarrow \uparrow \downarrow t & & \mathrm{fst} \downarrow \uparrow \downarrow \mathrm{snd} \\ (X/\!/\Gamma)_0 & := & X \end{array}$$

Points in set thy can't have symmetries, so we have to explicitly carry that data around.

Special features:

1) $s : (X/\!/\Gamma)_1 \to (X/\!/\Gamma)_0$ is étale.
2) $(s,t) : (X/\!/\Gamma)_1 \to (X/\!/\Gamma)_0^2$ is proper.

$\Big\}$ $X /\!/ \Gamma$ is proper étale.

Thm (Moerdijk-Pronk):

All orbifolds are presented by proper étale groupoids.

Orbifolds, synthetically:

Orbifolds are "smooth spaces" where the points have finite symmetries.

Points in HoTT have symmetries naturally

Working in Cohesive HoTT & Synthetic Differential Geometry:
crisp types internalize the external
infinitesimals give synthetic calculus

Standard in SDG; eg manifolds...

Def: An orbifold is a microlinear type whose types of identifications are properly finite.

discrete subquotients of finite sets.

Thm: The Rezk completion of a crisp*, ordinary*, proper étale pregroupoid is an orbifold.

* all ways of saying "the usual, external, proper étale groupoids"

Homotopy Type Theory is

- a logical system for working directly with sheaves of homotopy types.
- a standalone foundation of mathematics
  - Types A of mathematical objects
  - Elements $a : A$ of a given type. "a is an A"
  - Variable Elements $x^2+1 : \mathbb{R}$ (given that $x : \mathbb{R}$)

$$\underbrace{x : \mathbb{R}}_{\text{"Context"}} \vdash x^2+1 : \mathbb{R}$$

  - Variable types $M : \mathrm{Manifold},\ p : M \vdash T_p M : \mathrm{Vect}_{\mathbb{R}}$

$\mathbb{N}$ is the type of natural numbers
$\mathbb{R}$ is the type of real numbers
Set is the type of sets
$\mathrm{Vect}_{\mathbb{R}}$ is the type of real vector spaces
Type is the type of types.

[$x : A \vdash b(x) : B(x)$ means "$b(x)$ is a $B(x)$, given that x is an A"]

Pair Types:

$TM :\equiv (p : M) \times T_p M$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \times B(x) \qquad A \times B$$

is the type of pairs (a, b) with $a : A$ and $b : B(a)$.

Function Types:

$\mathrm{Vec}(M) :\equiv (p : M) \to T_p M$

- If $B(x)$ is a type for $x : A$, then

$$(x : A) \to B(x) \qquad A \to B$$

is the type of functions $x \mapsto f(x)$ where $x : A \vdash f(x) : B(x)$

$$
\begin{aligned}
(X : \mathbf{Type}) &\times ((x,\, y : X) \to (p,\, q : x = y) \to (p = q)) \\
&\times (\alpha : G \times X \to X) \\
&\times ((x : X) \to (\alpha(1, x) = x)) \\
&\times ((x : X) \to (g,\, h : G) \to (\alpha(gh, x) = \alpha(g, \alpha(h, x)))) \\
&\times \left((x,\, y : X) \to \begin{cases} ((g,p) : (g : G) \times (\alpha(g,x) = y)) \\ \times(((h,q) : (h : G) \times (\alpha(h,x) = y)) \to ((g,p) = (h,q))) \end{cases} \right) \\
&\times \|X\|
\end{aligned}
$$

Types of Identifications:

- If x and y are of type A, then

$$x \underset{A}{=} y \quad \text{is the type}$$

of ways to identify x with y as elements of A.

E.g.
- In $\mathrm{Vect}_{\mathbb{R}}$, $e : T_pM = \mathbb{R}^n$ is a linear isomorphism.
- In Manifold, $e : M = N$ is a diffeomorphism.
- In Type, $e : A = B$ is an equivalence.
- In $\mathbb{N}$, $n = m$ has a unique element if and only if n equals m.

"Univalence Axiom" of Voevodsky

Dictionary (Shulman, Lumsdaine, Kapulkin, Voevodsky, et al.)

Homotopy Type Theory	Sheaves of homotopy types
Type of object	Sheaf of homotopy types in $\mathcal{E}$
$x : A \vdash B(x) : \mathrm{Type}$	$B \xrightarrow{\pi} A$ in $\mathcal{E}/A$
$x : A \vdash b(x) : B(x)$	$A \xrightarrow{b} B \xrightarrow{\pi} A$ in $\mathcal{E}/A$
$(x : A) \times B(x)$	$B \to A \to *$ along $\mathcal{E}/A \xrightarrow{\Sigma_A} \mathcal{E}/*$
$(x : A) \to B(x)$	$\left\{ B \to A \right\}$ along $\mathcal{E}/A \xrightarrow{\Pi_A} \mathcal{E}/*$
$x, y : A \vdash (x = y) : \mathrm{Type}$	$PA \to A \times A$ The path space in $\mathcal{E}/A \times A$

Higher Groups via their Deloopings

Def(B-vD-R)*: A **higher group** is a type G identified with

the symmetries $(pt_{BG} = pt_{BG})$ $\quad\forall e : BG.\ \tau_{-1}(pt_{BG} = e)$

of a "canonical exemplar" $pt_{BG} : BG$ in a 0-connected type

E.g.

- $BGL_n(\mathbb{R})$ is the type of n-dim vector spaces.
  Canonical exemplar of $GL_n(\mathbb{R})$ is $\mathbb{R}^n$.
- $B\Sigma_n$ is the type of n-element sets.
  Canonical exemplar of Σ_n is $\underline{n} :\equiv \{0, \dots, n-1\}$.

Def(B-vD-R): A **homomorphism** $\varphi : G \to H$ is a pair

$$B\varphi : BG \to BH, \qquad pt_\varphi : B\varphi(pt_{BG}) = pt_{BH}$$

- The inclusion $\Sigma_n \hookrightarrow GL_n(\mathbb{R})$ is given by $F \mapsto \mathbb{R}^F$, $\quad \mathbb{R}^{\underline{n}} \doteq \mathbb{R}^n$.

* Buchholtz-van Doorn-Rijke

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Eg: The action α of the circle $U(1)$ on the plane $\mathbb{C}$ is

$$\begin{array}{ccccc}
B\alpha : & BU(1) & \longrightarrow & BAut(\mathbb{C}) & \\
& \| & & \| & \\
& \left\{\begin{array}{c}\text{1-dim} \\ \text{Hermitian} \\ \text{vector spc}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{types} \\ \text{identifiable} \\ \text{with } \mathbb{C}\end{array}\right\} & \\
& V & \longmapsto & V, & \text{the underlying set.}
\end{array}$$

$$pt_{B\alpha} :\equiv id$$

[An action α of G on X is a way $B\alpha$ of constructing types $B\alpha(e)$ from exemplars e of G.] !

Let's give this action:

C_2

We need to choose exemplars for C_2 which make this convenient...

$B\Sigma_2 :\equiv \left\{\begin{matrix}\text{2-element} \\ \text{sets}\end{matrix}\right\}$, then $e \longmapsto$???

$BO(1) :\equiv \left\{\begin{matrix}\text{1-dim} \\ \text{real inner} \\ \text{product spcs}\end{matrix}\right\}$, then $e \longmapsto \mathbb{R} \oplus e$

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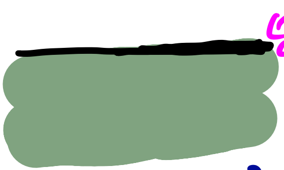
The quotient



$\circlearrowleft C_2$ is just the type of pairs!

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 /\!/_{\alpha} C_2 :\equiv (e : BO(1)) \times (\mathbb{R} \oplus e)$$

$$(x,y) \longmapsto (\mathbb{R}, (x,y))$$

Note: We have $-1 : (\mathbb{R}, (x,y)) = (\mathbb{R}, (x,-y))$

Examples of Orbifolds

$\mathcal{M}_{1,1} :\equiv (V : BU(1)) \times \mathcal{L}attice(V)$ (the elliptic curve associated to (V, Λ) is V/Λ)

$X^n /\!/ n! :\equiv (F : BAut(n)) \times X^F$ Configuration space

$\mathbb{T}^4 /\!/ C_2 :\equiv (K : BGal(\mathbb{C} : \mathbb{R})) \times \{z : K \mid N_{K:\mathbb{R}}(z) = 1\}^4$ Kummer Surface

The universal cover of $\mathcal{M}_{1,1}$ is

$$\mathfrak{h} :\equiv \{c : \mathbb{C} \mid Im(z) > 0\} \longrightarrow \mathcal{M}_{1,1}$$
$$c \longmapsto (\mathbb{C}, \mathbb{Z} \oplus c\mathbb{Z})$$

Each fiber is a $SL_2(\mathbb{Z})$-torsor, proving that $\int \mathcal{M}_{1,1} = BSL_2(\mathbb{Z})$

Homotopy Theory of Orbifolds, briefly:

Def: (Shulman) A type X is discrete if any path $\gamma : \mathbb{R} \to X$ is constant.

The shape (Schreiber) $(-)^{\int} : X \to \int X$ is the initial map from X to a discrete type.

Thm: The shape of $\mathcal{M}_{1,1}$ is $BSL_2(\mathbb{Z})$ via

$$\underbrace{(V : BU(1)) \times \mathcal{L}attice(V)}_{\mathcal{M}_{1,1}} \longrightarrow \underbrace{(V : BGL_2(\mathbb{R})) \times \mathcal{L}attice(V) \times (\mathbb{R} = \Lambda^2 V)}_{BSL_2(\mathbb{Z})}$$
$$(V, \ell) \longmapsto (V, \ell, Im\langle , \rangle)$$

Proof sketch: The fiber over $(\mathbb{R}^2, \mathbb{Z}^2, dx \wedge dy)$ is identifiable with $SL_2(\mathbb{R})/U(1)$, and this may be identified with $\mathfrak{h}$, which is contractible. "$\int$-connected"

SDG, really quickly:

A field $\mathbb{R}$, the smooth reals.

Def (Penon): A number $x : \mathbb{R}$ is infinitesimal if it is not distinct from 0: $\neg\neg(x = 0)$. $\quad \mathbb{D} :\equiv \{x \mid \neg\neg(x=0)\}$.

Axioms: *(Some of them)*

- (Kock-Lawvere) Any function $f(\varepsilon)$ of a number $\varepsilon^2 = 0$ is linear.
- $\mathbb{D}$ is tiny: $X \mapsto X^{\mathbb{D}}$ has an external right adjoint. *we handle this w/ Shulman's crispness!*

Def (Bergeron): A type X is microlinear if for any square

$$\begin{tikzcd} V_1 \arrow[r] \arrow[d] & V_3 \arrow[d] \\ V_2 \arrow[r] & V_4 \end{tikzcd}$$

of infinitesimal varieties such that

$$\begin{tikzcd} \mathbb{R}^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^{V_3} \arrow[d] \\ \mathbb{R}^{V_2} \arrow[r] & \mathbb{R}^{V_1} \end{tikzcd},$$

then

$$\begin{tikzcd} X^{V_4} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{V_3} \arrow[d] \\ X^{V_2} \arrow[r] & X^{V_1} \end{tikzcd}.$$

... includes all manifolds.

The Crystaline Modality and Étale Maps.

Def: The crystaline modality $\Im$ is $\mathrm{Loc}_{\mathbb{D}}$. *Axiom* It's lex.

Def (Schreiber, Cherubini): A map $f : X \to Y$ is $\Im$-étale if

the naturality sq

$$\begin{tikzcd} X \arrow[r, "(-)^{\Im}"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "(-)^{\Im}"'] & \Im Y \end{tikzcd}$$

is a pullback.

Thm: Crisp maps between ordinary manifolds are étale if and only if they are local diffeomorphisms.

Thm: If $f : X \to Y$ is surjective and étale and X is microlinear, then so is Y.

Good orbifolds are microlinear:

If Γ is (crisply) discrete and acts on X,
Then $X \to X /\!\!/ \Gamma$ is an ∞-cover and therefore étale.
So, by

Thm: If $f : X \to Y$ is surjective and étale and X is microlinear, then so is Y.

So, if X is microlinear, so is $X /\!\!/ \Gamma$.

We get all étale groupoids with a descent theorem:

Thm: If $f :: X \to Y$ is crisp, surjective, and $f^* f$ is étale, then f is étale.

Things I haven't covered:

- Finiteness: Follows from the covering axiom of Bunge & Dubuc: If $\mathbb{R} = X \cup Y$, then $\bigcup \mathbb{R} = \text{interior}\, X \cup \text{interior}\, Y$.
  ↳ If K is Dubuc-Penon compact, then every Penon open cover admits a subfinite refinement.

- Tiny infinitesimals and form classifiers.

- Differential Cohomology:

$$
\begin{tikzcd}
 & B^k_\nabla \mathbb{R} \arrow[rr, "\text{curvature}"] \arrow[dr, "\text{Universal } \infty\text{-cover}"'] & & \Lambda^{k+1}_{cl} \arrow[dr, "\text{de Rham class}"] & \\
bB^k \mathbb{R} \arrow[ur, "\text{Forget Flatness}"] \arrow[dr, "\exp"'] & & B^k_\nabla U(1) \arrow[ur, "\text{Curvature}"] \arrow[dr, "\text{Top. Class}"] & & bB^{k+1} \mathbb{R} \\
 & bB^k U(1) \arrow[ur, "\text{Forget Flatness}"] \arrow[rr, "\beta"'] & & B^{k+1} \mathbb{Z} \arrow[ur, "\text{inclusion}"'] &
\end{tikzcd}
$$

Future Work

- Constructing connection classifiers using tiny infinitesimals.

$$B_\nabla G :\equiv (e : BG) \times \Lambda^1(T_{id}\mathrm{Aut}(e))$$

  and Chern-Weil theory.

- Deligne-Mumford stacks in synthetic algebraic geometry à la Blechschmidt?

- Proper orbifold cohomology à la Sati-Schreiber, using equivariant cohesion.

Thank You