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On LOCALIZATIONS VIA HOMOTOPIES

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Description of some work in the context of Localizations of a category $\mathcal{C}$ at a class $\mathcal{W}$ of arrows "$\xrightarrow{\ \circ\ }$"

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[dr, "F"'] & Ho(\mathcal{C}) \arrow[d, dashed, "\tilde{F}"] \\ & \mathcal{D} \end{tikzcd}$$

commutes in a appropriate way.

$\xrightarrow{F}$ sends w.e. to equivalences

$\overset{\tilde{F}}{\dashrightarrow}$ $\left\{ \right.$ a] Categories and Functors
b] 2-Categories and 2 Functors
c] Simplicial Categories and Simplicial Functors

$\boxed{\text{Objects of } Ho(\mathcal{C}) \text{ are the same that of } \mathcal{C}}$

a] 1) gabriel-Zisman $X \to \cdot \xleftarrow{\circ} \cdot \to \cdot \xleftarrow{\circ} \cdot \cdots \to \cdot \xleftarrow{\circ} Y$

Calculus of fractions $X \to \cdot \xleftarrow{\circ} Y$

2) Quillen model categories

Define homotopies between arrows which determine a congruence on $\mathcal{C}$ and take the quotient category (equivalent classes under homotopy)

$\boxed{\text{Localization}}$

b]

Dubuc-girabel based on work by Descotte-Dubuc-Szyld

Same arrows than $\mathcal{C}$, take homotopies as 2-cells (actually under an equivalence relation)

(All 2-cells are invertible)

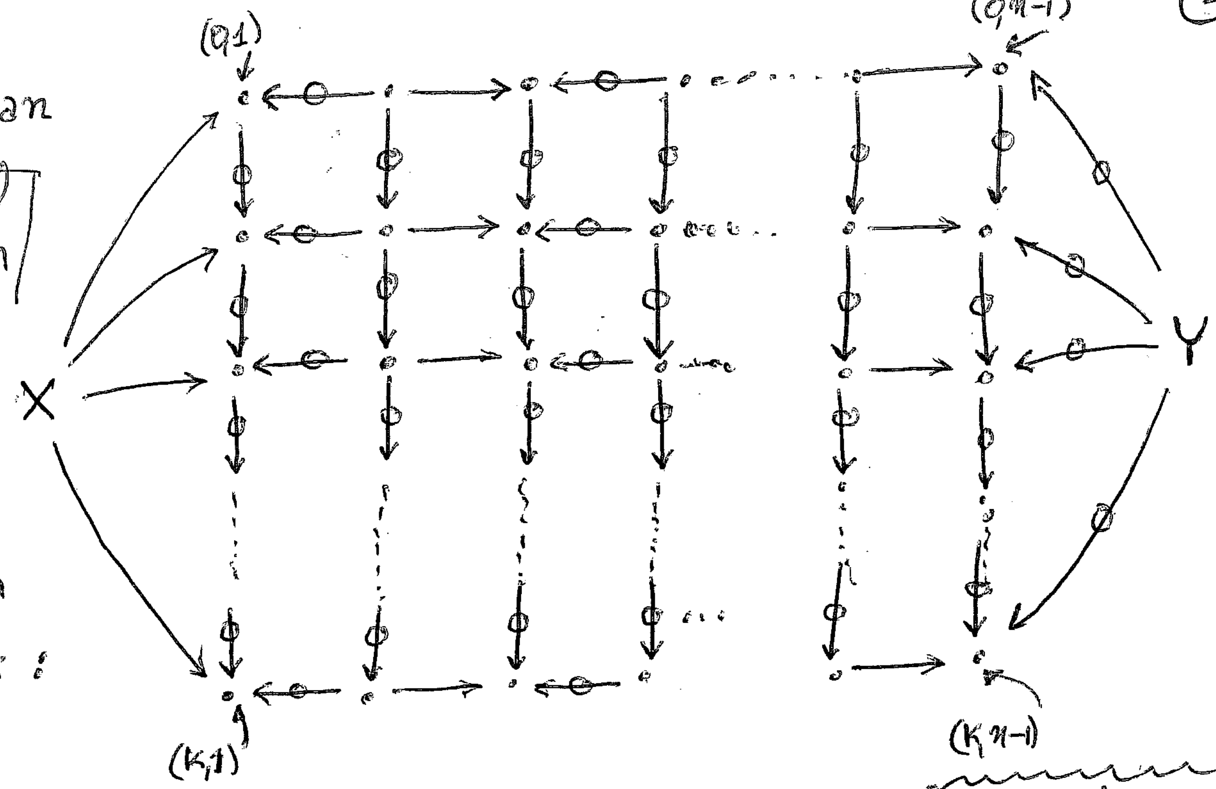
$\boxed{\text{2-Localization}}$

c] Dwyer-Kan

Simplicial Localization

Hammoc construction

K-Simplices :



For details have to define reduced hammoc

Faces are obtained by omitting a row
Degeneracies are obtained by repeating a row.

A different context is when $\mathcal{C}$ is not just a category -

Case where $\mathcal{C}$ is a 2-category

1) Pronk - 2-calculus of fractions (corresponds to a] 1))

2) Descotte-Dubuc-Szyld Model Bicategories (corresponds to a] 2))

Are very different theories because of the presence of non-invertible 2-cells.

In this talk I will consider the case b] of 2-localization

All complete proof are in the arXiv paper 2208.00314v1

Dubuc-Girabel : The 2-localization of a model category

Category $\mathcal{C}$ and a class of arrows $\mathcal{W} \subset \mathcal{C}$ such that

$id_X \in \mathcal{W}$,
$$\begin{tikzcd} & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd}$$
two $\in \mathcal{W} \Rightarrow$ third $\in \mathcal{W}$

2-Localization of $\mathcal{C}$ at $\mathcal{W}$ $\quad \mathcal{C} \xrightarrow{\;i\;} H_o(\mathcal{C})$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[dr, "F\ \square"'] & Ho\,\mathcal{C} \arrow[d, dashed, "\tilde{F}"] \\ & \mathcal{D} \end{tikzcd}$$

(L) $\begin{cases} \xrightarrow{\;i\;\diamond\;} \\ Hom(H_o(\mathcal{C}), \mathcal{D}) \xrightarrow{\;i^*\;} Hom(\mathcal{C}, \mathcal{D})_+ \end{cases}$

(PL) $Hom(H_o(\mathcal{C}), \mathcal{D})_+ \xrightarrow{\;i^*\;} Hom(\mathcal{C}, \mathcal{D})_+$

$\boxed{\text{Clearly } \xrightarrow{\;i\;\diamond\;} + \text{(PL)} \Rightarrow \text{(L)}}$

Hom is the 2 category of 2-functors
<u>pseudonatural</u> transformations
i^* (pre-composing with i) is a
<u>pseudo equivalence</u> of 2-categories

I] a) We construct a 2-category $Ho(\mathcal{C})$ using <u>cylinders</u> (more general than Quillen's) and the corresponding homotopies, which satisfies (PL), actually with i^* a isomorphism.

b) We show that split weak equivalences are equivalences in $Ho(\mathcal{C})$, thus if $\mathcal{W}$ is split-generated, (L) holds

a) Definition of cylinder and homotopy:

$$C = \begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[dr, "x"'] \arrow[rr, bend left=40, "f"] \arrow[rr, bend left=25, "g"'] & W \arrow[d, "s"] \arrow[r, "h"] \arrow[dl, phantom, "///"] & Y \\ & Z & \end{tikzcd}$$

$hd_0 = f$

$hd_1 = g$

Cylinder C

Homotopy $H = (C, h)$

$H : f \rightsquigarrow g$

given $\boxed{\mathcal{C} \xrightarrow{F} \mathcal{D} \text{ and } H : f \rightsquigarrow g}$ we have

$$\begin{tikzcd} FX \arrow[r, shift left, "Fd_1"] \arrow[r, shift right, "Fd_0"'] \arrow[dr, "Fx"'] & FW \arrow[d, "Fs"] \arrow[r, "Fh"] \arrow[dl, phantom, "///"] & Y \\ & FZ & \end{tikzcd}$$

Fs a equivalence

full & faithful

$$\mathcal{D}(FX, FW) \xrightarrow{(Fs)_*} \mathcal{D}(FX, FZ)$$

$\exists! \; \widehat{FC} : Fd_0 \Rightarrow Fd_1$ UNIQUE $\;/\;$ $Fs\, \widehat{FC} = Fx$

Define $\boxed{\widehat{FH} : Ff \Rightarrow Fg \,, \quad \widehat{FH} = Fh\, \widehat{FC}}$

Equivalence relation between homotopies called "ad hoc":

$$H \sim K \iff_{def} \forall\, \mathcal{C} \xrightarrow{F} \mathcal{D}, \quad \widehat{FH} = \widehat{FK}$$

More generally, given sequences of composable homotopies

$$\sim\left\{\begin{array}{l}(H_n,\dots H_2,H_1): f \overset{H_1}{\rightsquigarrow} r_1 \overset{H_2}{\rightsquigarrow} r_2 \rightsquigarrow \cdots\cdots r_{n-1} \overset{H_n}{\rightsquigarrow} g \\ (K_m,\dots K_2,K_1): f \overset{K_1}{\rightsquigarrow} s_1 \overset{K_2}{\rightsquigarrow} s_2 \rightsquigarrow \cdots \quad s_{m-1} \overset{K_m}{\rightsquigarrow} g\end{array}\right.$$

$$\underset{def}{\Longleftrightarrow} \quad \widehat{FH_n}\circ\cdots\widehat{FH_2}\circ\widehat{FH_1} = \widehat{FK_m}\circ\cdots\widehat{FK_2}\circ\widehat{FK_1} : Ff \Rightarrow Fg$$

__Def__ For each pair of objects X, Y define a category $Ho(\mathcal{C})(X,Y)$ with objects the arrows in $\mathcal{C}$ and arrows the equivalent classes of composable homotopies, composed by juxtaposition.

__Illustration__ ① given two composable homotopies $\overbrace{}^{H_1,\, H_2}$, they __compose__ if exists K such that $(H_2, H_1) \sim (K)$

In this case $[H_2]\circ[H_1] = [K]$

② given an arrow f, a homotopy $f \overset{H}{\rightsquigarrow} f$ determine the identity $[H] = id_f$ if and only if $\widehat{FH} = id_{Ff}$,

For example, the following homotopies determine id_f.

$$\left|\begin{tikzcd} X \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] \arrow[dr, "id"'] & X \arrow[r, "f"] \arrow[d, "id"] & Y \\ & X & \end{tikzcd}\right| \sim \left|\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] \arrow[dr, "f"'] & Y \arrow[r, "id"] \arrow[d, "id"] & Y \\ & Y & \end{tikzcd}\right|$$

③ All 2-cells of $Ho(\mathcal{C})$ are invertible.
given C and $\boxed{H=(C,h): f \rightsquigarrow g}$, define C^{-1} by switching d_0 and d_1, and set $\boxed{H^{-1}=(C^{-1},h): g \rightsquigarrow f}$.
It is easy to check $[H]^{-1}=[H^{-1}]$.

The horizontal composition follows from the wiskerings $\xrightarrow{l} \overset{H}{\rightsquigarrow} \xrightarrow{r}$

$$\begin{tikzcd}
X' \arrow[r, "l"] \arrow[drr, bend right, "zl"'] & X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rr, bend left=60, "f"] \arrow[rr, bend left=35, "g"'] \arrow[dr, "z"] & W \arrow[r, "h"] \arrow[d, "s"] & Y \arrow[r, "r"] & Y' \\
 & & Z & &
\end{tikzcd}$$

$H=(C,h): f \rightsquigarrow g$

$$\boxed{\begin{array}{ll} Hl=(Cl,h): fl \rightsquigarrow gl & (\text{same } h) \\ rH=(C,rh): rf \rightsquigarrow rg & (\text{same } C) \end{array}}$$

It is easy to check ⊛ $\widehat{F(Hl)} = \widehat{FH}\,Fl$, $\widehat{F(rH)} = Fr\,\widehat{FH}$

Define $\boxed{[H]l=[Hl] \text{ and } r[H]=[rH]}$

Rather straightforward to show that this is well-defined.
The wiskering axioms follow by applying F and the axioms of $\mathcal{D}$.

We have a 2-category $Ho(\mathcal{C})$ and a 2-functor $\mathcal{C} \xrightarrow{i} Ho(\mathcal{C})$, the identity on objects and arrows

Universal Property

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[dr, "F"'] \arrow[dr, phantom, shift left=3, "\cong"] & \widetilde{Ho}(\mathcal{A}) \arrow[d, dashed, "\exists! \tilde{F}"] \\ & \mathcal{D} \end{tikzcd}$$

$$\tilde{F}X = FX \quad , \quad \tilde{F}f = Ff \quad , \quad \tilde{F}([H]) = \widehat{FH}$$

in general $F([H_n, \dots H_2, H_1]) = \widehat{FH_n} \circ \dots \widehat{FH_2} \circ \widehat{FH_1}$

By definition of the ad-hoc relation this is well-defined

Uniqueness hold (not easy).

Can also prove the 2-categorical aspect:

given $F \overset{\eta}{\Longrightarrow} G$ define $\tilde{F} \overset{\tilde{\eta}}{\Longrightarrow} \tilde{G}$

$FX \xrightarrow{\eta X} GX$. $\tilde{F}X \xrightarrow{\tilde{\eta}X} \tilde{G}X$

$$\boxed{\tilde{\eta}X = \eta X}$$

Conditions of 2-naturality for $\tilde{\eta}$ hold (not easy)

This finishes the proof the principal theorem.

Theorem: $\boxed{\mathcal{C} \xrightarrow{i} Ho(\mathcal{C}) \text{ satisfies (PL) in such a way that } i^* \text{ is an isomorphism of 2-categories}}$

From the 3×2 property and the fact that any cylinder C determines an invertible 2-cell $d_0 \Rightarrow d_1$ it follows that any split weak equivalence becomes an equivalence in $H_0(\mathcal{C})$. Thus:

CONDITION (SG). $\mathcal{W}$ is split generated

Theorem If $\mathcal{W}$ is split generated, then $\mathcal{C} \xrightarrow{i} Ho(\mathcal{C})$ satisfies (L) in such a way that i^* is an isomorphism of 2-categories

$$\boxed{\begin{array}{c} Hom(Ho(\mathcal{C}_{fc}), \mathcal{D}) \xrightarrow{i^*} Hom(\mathcal{C}, \mathcal{D})_+ \\ \text{is an isomorphism of 2-categories.} \end{array}}$$

II] Model Category. Three classes of arrows, $\mathcal{W}$, $\mathcal{F}$: fibrations "$\twoheadrightarrow$" $\mathcal{C}of$ cofibrations "$\rightarrowtail$"

$\mathcal{W}$ does not satisfy (SG). Define a full subcategory $\mathcal{C}_{fc} \subset \mathcal{C}$ of fibrant-cofibrant objects such $\mathcal{W}|_{\mathcal{C}_{fc}}$ satisfy (SG), and a fibrant-cofibrant <u>replacement</u>

$$\underset{\text{cofibrant}}{X \xleftarrow[\circ]{p_X} QX}, \quad \underset{\text{fibrant}}{X \xrightarrow[\circ]{\nu_X} RX}, \quad X \xleftarrow[\circ]{p_X} QX \xrightarrow[\circ]{\nu_{QX}} \underset{\text{fibrant-cofibrant}}{RQX}$$

Then work hard and show that $Ho(\mathcal{C}_{fc})$ is the 2-localization of $\mathcal{C}$.

Three facts that are the essence(*) of the concept of Model category are the following:

The <u>dual category</u> of a Model category is a Model category

FX	Factorization Axiom
LX	Lifting Axiom

(*) <u>essence</u>: the intrinsic nature or indispensable quality of something, especially something abstract, which determines its character. (Oxford Dictionary).

LX $\boxed{\begin{tikzcd} \bullet \arrow[r] \arrow[d, tail] & \bullet \arrow[d, two heads, "\circ"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd} \text{ and } \begin{tikzcd} \bullet \arrow[r] \arrow[d, tail, "\circ"'] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}}$ FX $\boxed{\begin{tikzcd} \bullet \arrow[rr] \arrow[dr, tail, "\circ"'] & & \bullet \\ & \bullet \arrow[ur, two heads] & \end{tikzcd} \text{ and } \begin{tikzcd} \bullet \arrow[rr] \arrow[dr, tail] & & \bullet \\ & \bullet \arrow[ur, two heads, "\circ"'] & \end{tikzcd}}$

Def X: fibrant $X \twoheadrightarrow 1$, cofibrant $0 \rightarrowtail X$

Prop If X is fibrant and Y is cofibrant, then any $X \xrightarrow[\circ]{f} Y$ factors as a section followed by a retraction.

proof

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, tail, "i"', "\circ"] & X \arrow[d, two heads] \\ Z \arrow[r] \arrow[ur, dashed] & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d, tail] & Z \arrow[d, two heads, "p", "\circ"'] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f", "\circ"'] \arrow[dr, tail, "i", "\circ"'] \arrow[loop left, "id"] & & Y \arrow[dl, dashed, bend left] \arrow[loop right, "id"] \\ & Z \arrow[ul, dashed, bend left] \arrow[ur, two heads, "p", "\circ"'] & \end{tikzcd}$$

If X and Y are f.c., then Z is f.c.

Corollary $\mathcal{C}_{fc} \xrightarrow[\circ]{i} Ho(\mathcal{C}_{fc})$ is the 2-localization

$$Hom(Ho(\mathcal{C}_{fc}), \mathcal{D}) \xrightarrow{i^*} Hom(\mathcal{C}_{fc}, \mathcal{D})$$

is a isomorphism of 2-categories

(Remark that $\mathcal{C}_{fc}$ is not a model category)

Cofibrant replacement Q on objects and arrows:

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr, hook] & X \\ & QX \arrow[u, "p_X"'] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[rr] \arrow[d] & & QY \arrow[d, "p_Y"] \\ QX \arrow[urr, dashed, "Qf"] \arrow[r, "p_X"'] & X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \boxed{\begin{tikzcd} \arrow[r, "f"] \arrow[d, phantom, "\Downarrow"] & {} \\ \arrow[r, "Qf"] & {} \end{tikzcd}}$$

It is not a functor, $Q(gf)$ not equal to $Q(g)Q(f)$.

Easy solution: Functorial factorizations

$\lambda f \in \mathbb{L}$
$\rho_f \in \mathbb{R}$
S

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\lambda_f"'] & & \cdot \\ & \cdot \arrow[ur, "\rho_f"'] & \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[rr, bend left, "f"] \arrow[r, "\lambda_f"] \arrow[d, "u"'] & \cdot \arrow[r, "\rho_f"] \arrow[d, "S(u,v)"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[rr, bend right, "g"'] \arrow[r, "\lambda_g"'] & \cdot \arrow[r, "\rho_g"'] & \cdot \end{tikzcd}$$

FFX (Functorial factorization axiom)

$\mathbb{L} = \mathcal{W} \cap c\mathcal{F}$, $\mathbb{R} = \mathcal{F}$ and $\mathbb{L} = c\mathcal{F}$, $\mathbb{R} = \mathcal{W} \cap \mathcal{F}$

are functorial factorization systems

Immediate: Q is a functor $\mathcal{C} \longrightarrow \mathcal{C}$ in such a way that p_X becomes a natural transformation $Q \Rightarrow id$.

By duality there is also a fibrant replacement R functor, and combining both we have

$$\boxed{id \xLeftarrow{p} Q \xRightarrow{vQ} RQ, \quad X \xleftarrow{p_X} QX \xrightarrow{v_{QX}} RQX}$$

The localization result follows from this

Set $q = ir$,
$$\begin{tikzcd} & \mathcal{C}_{fc} \arrow[dr, "i"] & \\ \mathcal{C} \arrow[ur, "r"] \arrow[rr, "q"', "\diamond"] & & Ho(\mathcal{C}_{fc}) \end{tikzcd}$$

q is the 2-localization of $\mathcal{C}$ at $\mathcal{W}$.

$$Hom(Ho(\mathcal{C}_{fc}), \mathcal{D}) \xrightarrow{q^*} Hom(\mathcal{C}, \mathcal{D})_+$$

is a pseudoequivalence of 2-categories.

<u>proof</u> Consider
$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[d, "i"'] \arrow[dr, "q"] & \mathcal{C}_{fc} \arrow[d, "i", "\diamond"'] \\ Ho(\mathcal{C}) \arrow[r, dashed, shift left, "\bar{r}"] & Ho(\mathcal{C}_{fc}) \arrow[l, shift left, "\bar{J}"] \end{tikzcd}$$

$\boxed{\mathcal{C}_{fc} \overset{j}{\hookrightarrow} \mathcal{C} \quad rj = id}$

$\bar{J}$ by (L)

$\bar{r}$ by (PL)

$q = \bar{r} i$

$q^* = i^* \bar{r}^*$. By (PL) i^* is even a isomorphism

Remains to see that $\bar{r}^*$ is a pseudoequivalence.

$$Hom(Ho(\mathcal{C}_{fc}), \mathcal{D}) \underset{\bar{J}^*}{\overset{\bar{r}^*}{\rightleftarrows}} Hom(Ho(\mathcal{C}), \mathcal{D})_+$$

$\boxed{\bar{J}^* \bar{r}^* = (\bar{r}\bar{J})^* = id^* = id}$. Observe $\bar{J}\bar{r}X = RQX$

Then, for $Ho(\mathcal{C}) \xrightarrow{F} \mathcal{D}$, $(\bar{r}^* \bar{J}^* F)X = ((\bar{J}\bar{r})^* F)X =$

$= (F\bar{J}\bar{r})X = FRQX \underset{\sim}{\xleftarrow{F r_{QX}}} FQX \underset{\sim}{\xrightarrow{F p_X}} X$

This is the idea of the proof, details <u>not</u> straight forward. ∎

III] RECOVERING QUILLEN

Def A Quillen cylinder is a cylinder where $Z = X$, $x = id_X$ and $X \amalg X \xrightarrow{\binom{d_0}{d_1}} W$ is a cofibration.

A Quillen homotopy is an homotopy with a Quillen cylinder

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend left=30, "g"] \arrow[dr, "id"'] & W \arrow[r, "k"] \arrow[d, "s"] & Y \\
& X &
\end{tikzcd}$$

Easy: if X is cofibrant, d_0 and d_1 are $\rightarrowtail\!\circ\!\rightarrow$

Prop. Quillen homotopies compose.

$$\begin{tikzcd}
X \arrow[dr, tail, "d_0"] \arrow[ddrrr, bend left, "f"] & & & \\
& W \arrow[dr, tail, "\alpha_0"] \arrow[drr, bend left, "h"] \arrow[ddddr, "s"] & & \\
X \arrow[ur, tail, "d_1"] \arrow[dr, tail, "d_0'"] \arrow[r, phantom, "po"] & & W'' \arrow[r, dashed, "h''"] \arrow[ddd, dashed, "s''"] & Y \\
& W' \arrow[ur, tail, "\beta_0"] \arrow[urr, bend right, "h'"] \arrow[ddr, "s'"] & & \\
X \arrow[ur, tail, "d_1'"] \arrow[uurrr, bend right, "t"] & & & \\
& & X &
\end{tikzcd}$$

$(c, h): f \rightsquigarrow g$

$(c'h'): g \rightsquigarrow t$

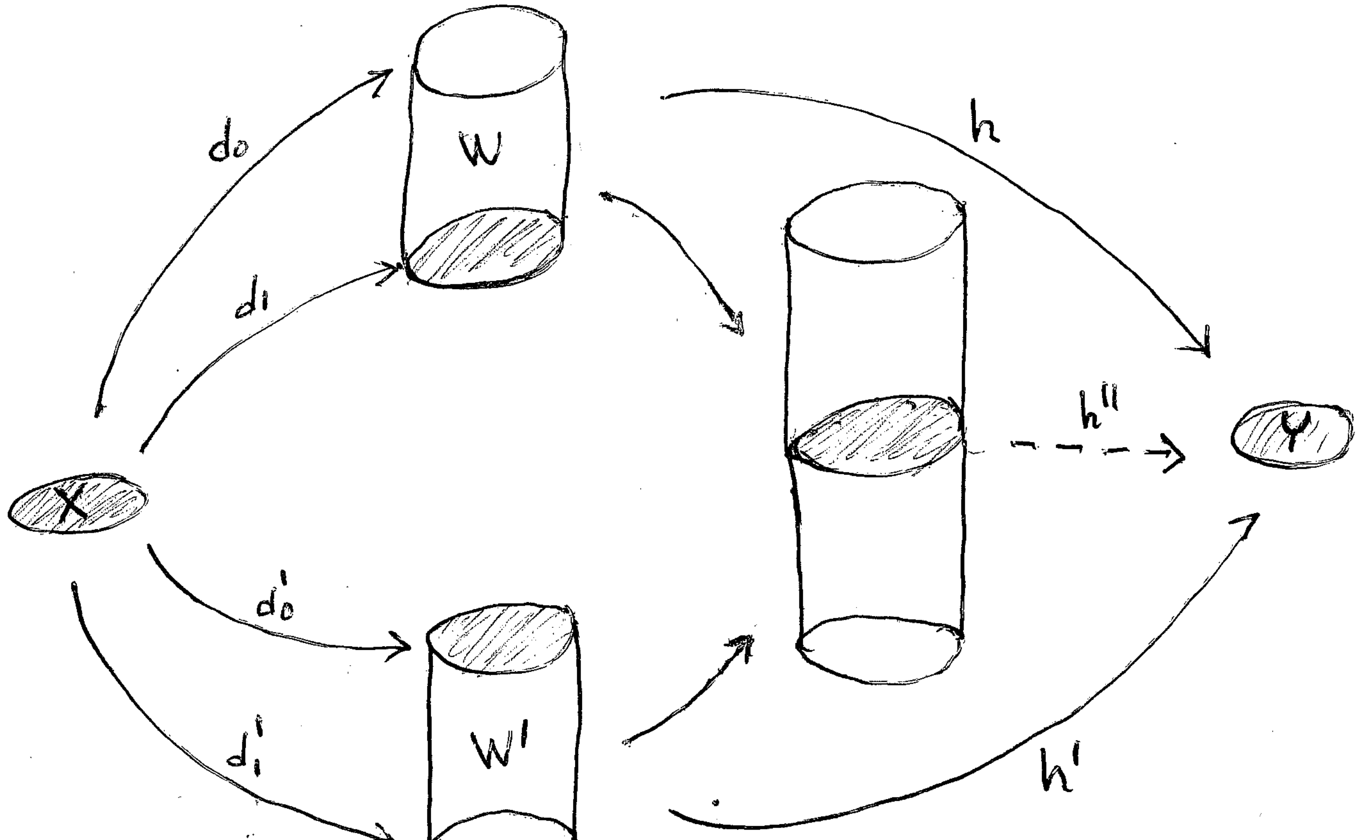
get

$H'': f \rightsquigarrow t$

$(H'H) \sim H''$

thus $[H'] \circ [H] = [H'']$

In the category of topological spaces this corresponds to gluing the bottom of the first cylinder with the top of the second:



Computing with the axioms it readily follows

Prop.

X, Y fibrant-cofibrant $\qquad X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$H = (C, h) : f \rightsquigarrow g$

Then $\exists$ $H' = (C', h') : f \rightsquigarrow g$

with C' a Quillen cylinder in $\mathcal{C}_{cf}$

and $H' \sim H$, $[H'] = [H]$

Horizontal composition of Quillen's homotopies is defined in $\mathcal{C}_{fc}$, not in $\mathcal{C}$.

$$
\begin{tikzcd}
X' \arrow[r, "l"] \arrow[drr, bend right, "l"'] & X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend left=30, "g"'] \arrow[dr, "id"] & W \arrow[r, "h"] \arrow[d, "s"] & Y \arrow[r, "r"] & Y' \\
 & & X & &
\end{tikzcd}
$$

$H = (C, h) : f \rightsquigarrow g$

$rH = (C, rh) : rf \rightsquigarrow rg$ no problem (same cylinder)

$Hl = (Cl, h) : fl \rightsquigarrow gl$, Cl is not a Quillen cylinder

Take K a Quillen homotopy such that $[K] = [Hl]$

Then $[H]l = [Hl] = [K]$. We have:

> The homotopy 2-category $Ho(\mathcal{C}_{cf})$ coincides with the one whose 2-cells are classes of a single Quillen homotopy

END

comment: Quillen defines $[H]l = [K]l = [Kl]$ where K is a right homotopy such that $[K] = [H]$