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Electrical Circuits with String Diagrams

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My takeaways from David Spivak's talk "What are we tracking?"

- We have the tools to tackle the age of complexity!
  - compositionality
  - structure / mathematical design patterns
  - cool diagrammatic syntax
  - better treatment of corner cases
  - **mathematics of open systems**

Graphical Relational Algebras

strict symmetric monoidal cats, usually props

Syntax ⟶ **Semantics**

- symmetric monoidal theories
- string diagrams as syntax
- diagrammatic reasoning
- graphical relational algebra

Relations

Linear Relations

Additive Relations

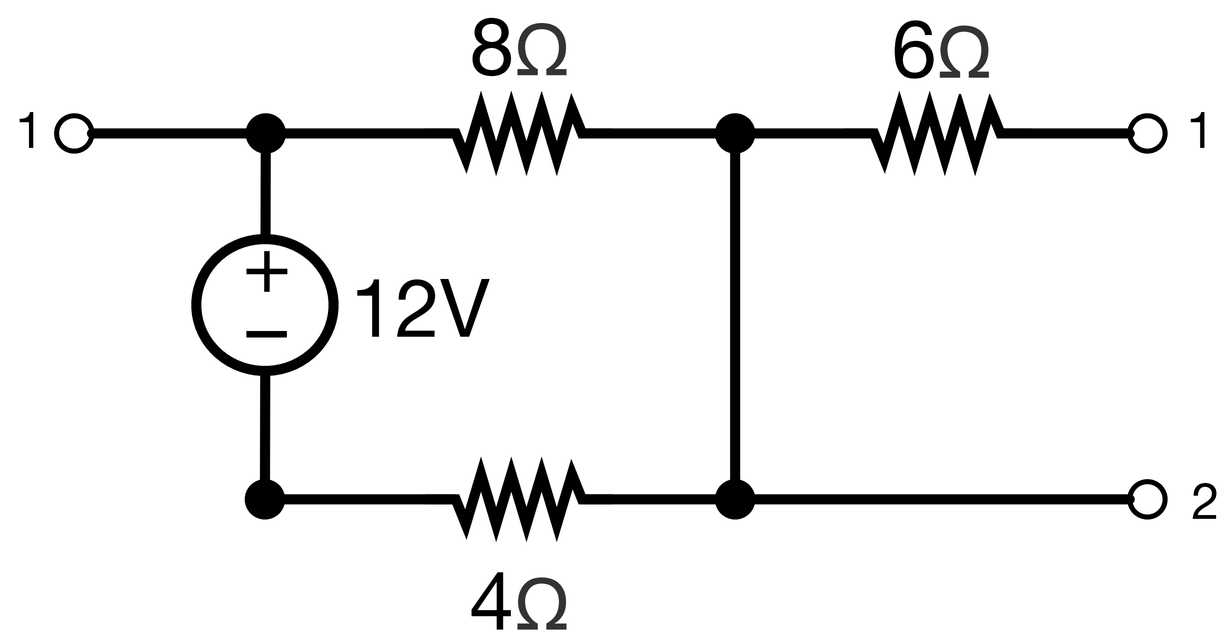
Affine Relations

"Stateful" Relations

Polyhedral Relations

Piecewise-Linear Relations

Mathematics of Open Systems



- I want the diagram above to be **first class syntax**
- I want a useful **calculus**: (in)equational characterisations of semantic identity
- formally, an arrow of a monoidal category (prop)
- but with *relational* semantics instead of *functional* semantics

Plan

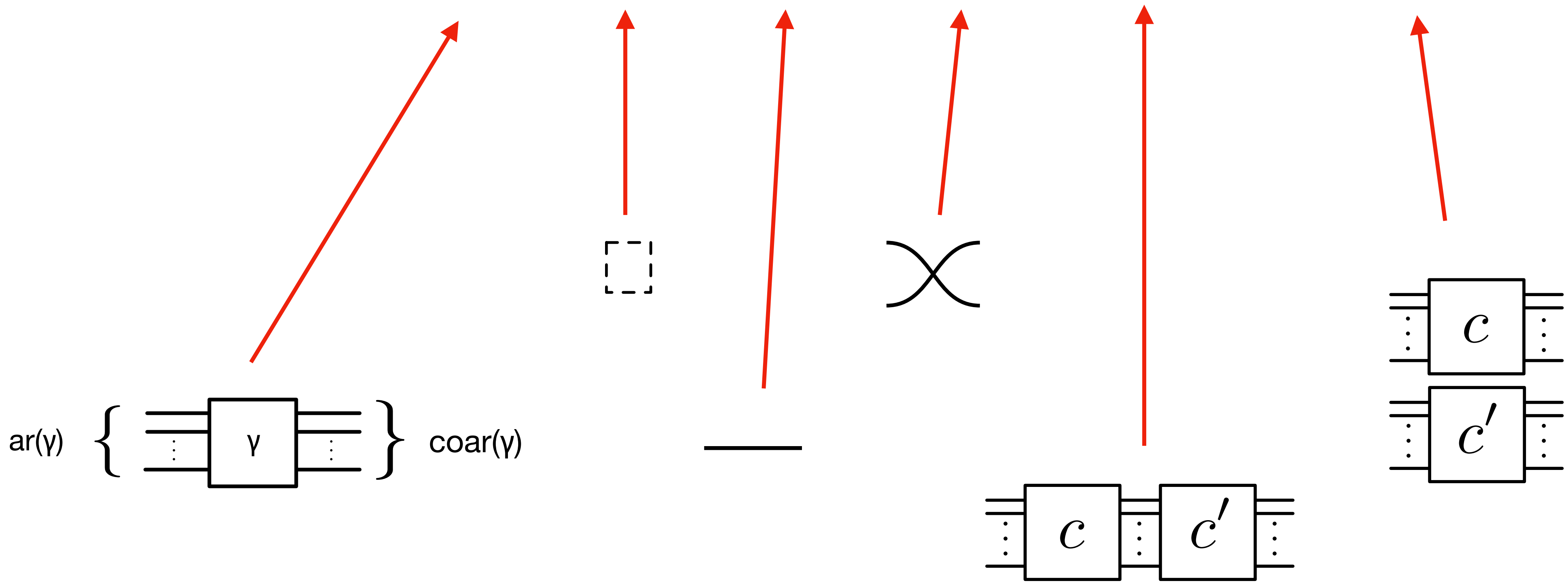
- **String diagrams**
- Universal algebra with string diagrams
- Graphical linear algebra
- Graphical affine algebra and electrical circuits

Presenting symmetric monoidal categories

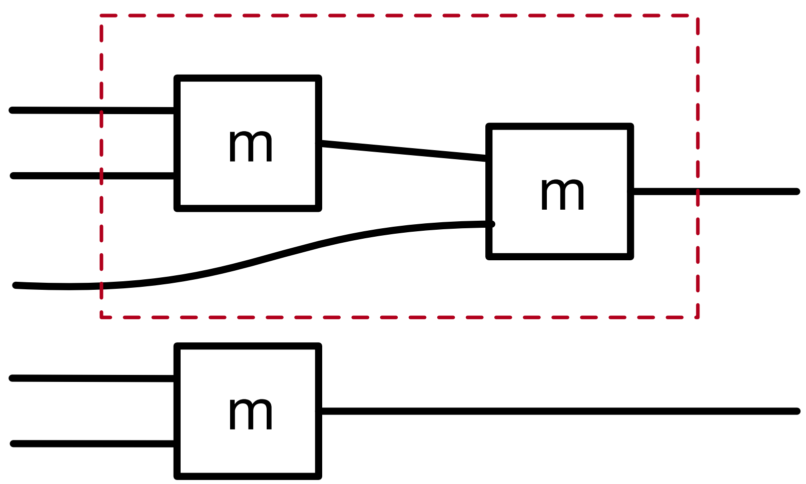
- Monoidal signature
  - $\Gamma = \{ \gamma : (ar(\gamma), coar(\gamma)) \}$
    - $ar(\gamma) \in \mathbf{N}$ - **arity** of γ
    - $coar(\gamma) \in \mathbf{N}$ - **coarity** of γ
- Term syntax for arrows
  - $c, c' \;::=\; \gamma \;|\; \varepsilon \;|\; id \;|\; \sigma \;|\; c \,;\, c \;|\; c \otimes c$

Diagrammatic conventions

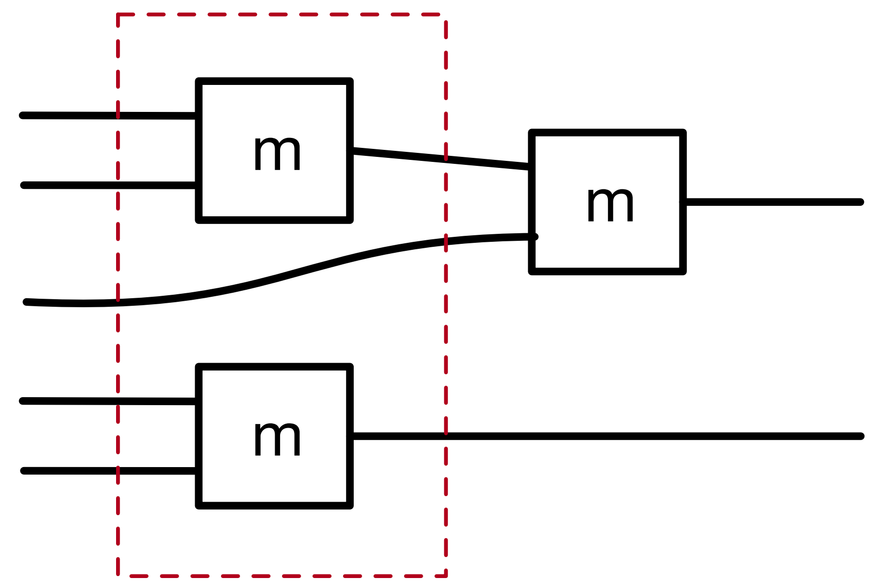
c,c’ ::= γ | ε | id | σ | c ; c’ | c ⊗ c’



Constructing diagrams



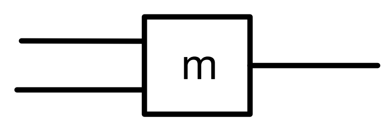
((m ⊗ id) ; m) ⊗ m



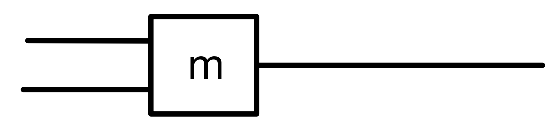
(m ⊗ id ⊗ m) ; (m ⊗ id)

- To disambiguate terms one would need to introduce additional “parentheses” boxes

Only connectivity matters



m



m ; id

- It also happens that different terms lead to diagrams with the same connectivity

Fundamental theorem

Theorem: Two diagrams obtained from terms c, c’ have the same connectivity iff the terms are equated by the theory of symmetric strict monoidal categories.

String diagram = class of diagrams obtained from a term, up-to “only connectivity matters”

In particular: string diagrams are the arrows of the **free symmetric strict monoidal category on Γ**

objects = natural numbers (“dangling wires”)

arrows = string diagrams

Plan

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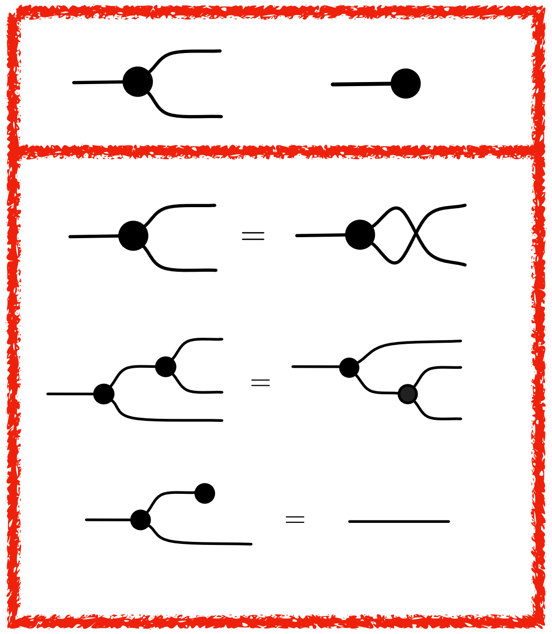
Symmetric monoidal theories

- A presentation of a symmetric monoidal theory is a pair (Γ, E) where
  - Γ is a monoidal signature
  - E is a set of pairs of string diagrams



- Example: Commutative comonoids

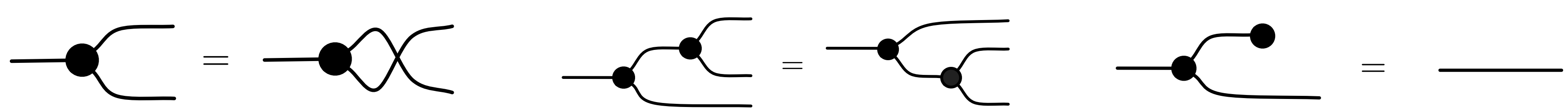
- Any presentation yields a symmetric monoidal category
  - arrows are string diagrams modulo “string diagram surgery” or “diagrammatic reasoning”

Cartesian categories

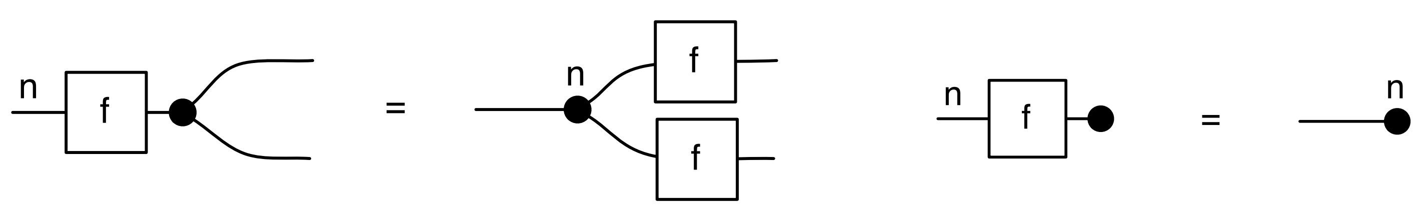
(Fox 1976)

cartesian categories are those sym. mon. cats. where every object has

commutative comonoid structure



and everything commutes with the structure

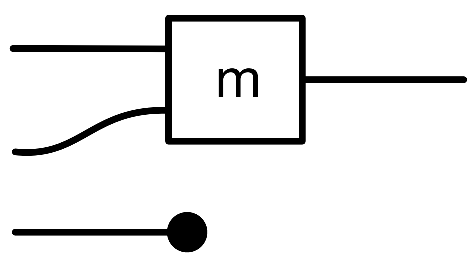


Example: **Set**$_\times$

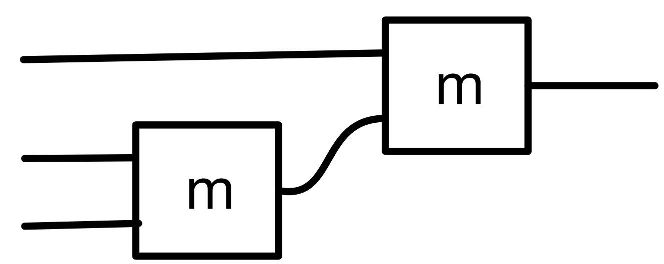
Classical terms vs string diagrams

- consider the theory of magmas, one binary operation m

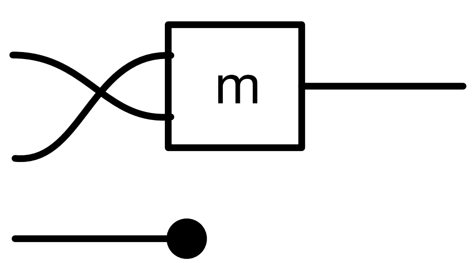
x,y,z ⊢ m(x,y)



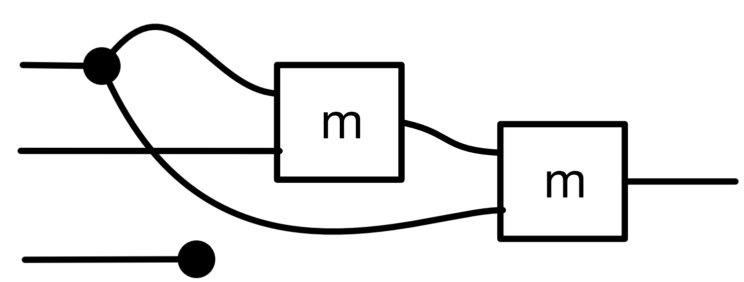
x,y,z ⊢ m(x,m(y,z))



x,y,z ⊢ m(y,x)

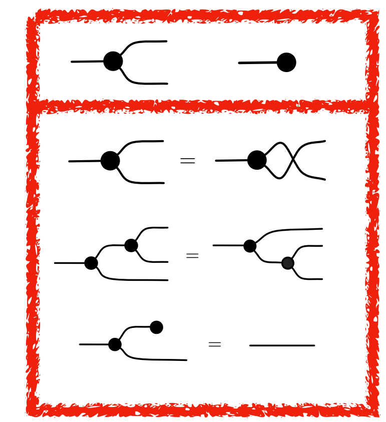


x,y,z ⊢ m(m(x,y),x))



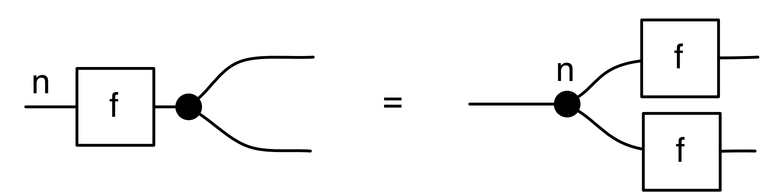
Lawvere theories

- Lawvere theory = cartesian prop

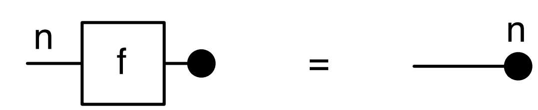


- recipe for Lawvere-theories-as-props

  1. add a cocommutative comonoid structure

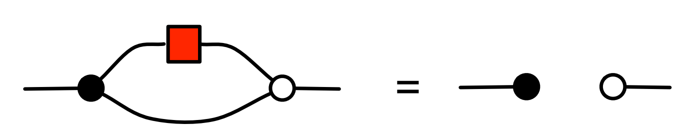


  2. make all generators commute with it



  3. add your other equations (which may make use of the comonoid structure)

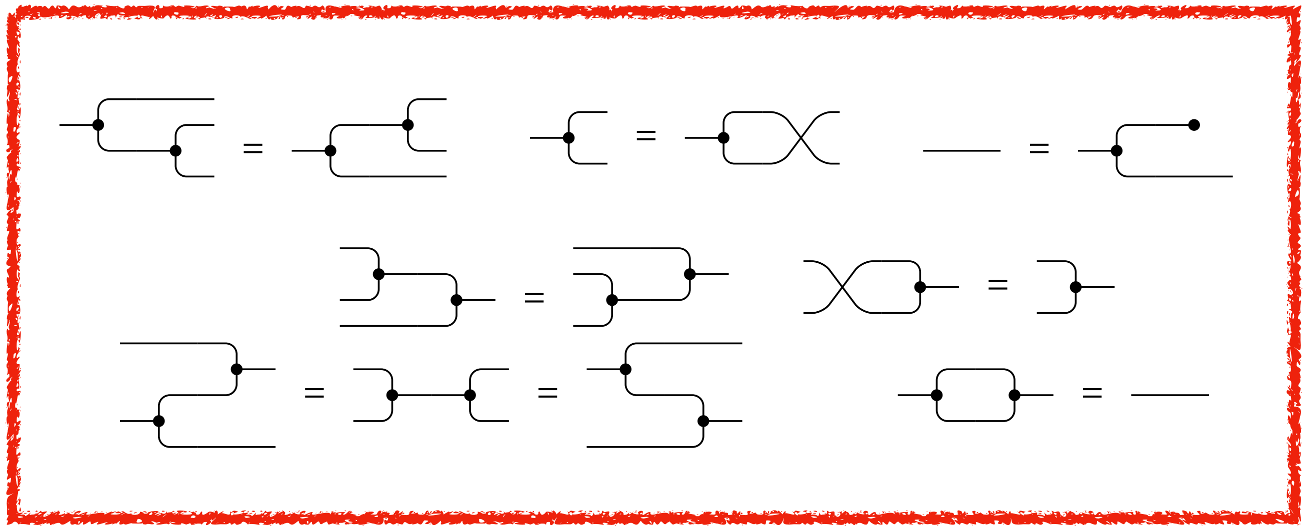
e.g. $x \cdot x^{-1} = e$



Partial theories

(Di Liberti, Loregian, Nester, S. 2021)

- Partial theory = discrete cartesian restriction prop
- recipe for partial-as-locally-ordered-props
  - add a partial Frobenius structure
  - make all your generators commute with **comultiplication**
  - add your other equations (which may make use of the partial Frobenius structure)

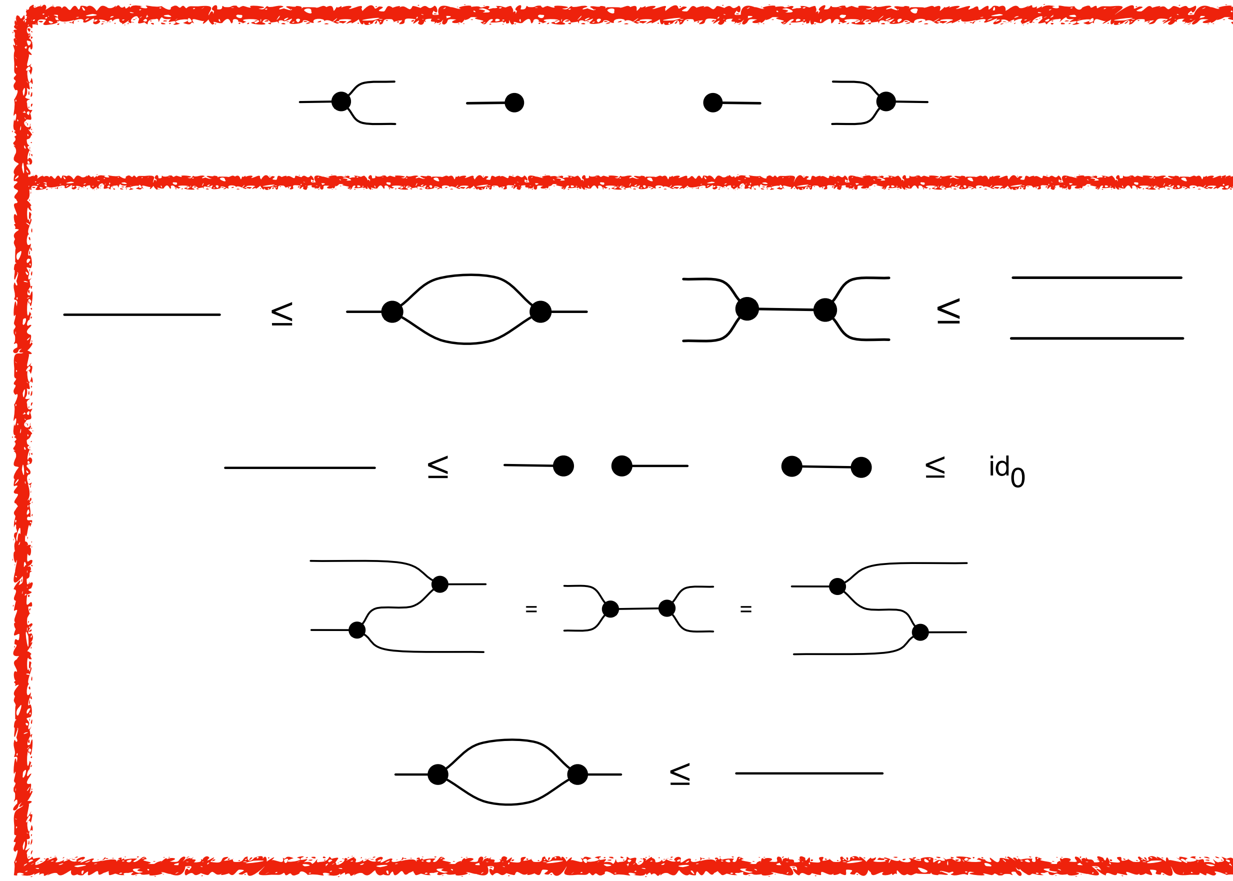


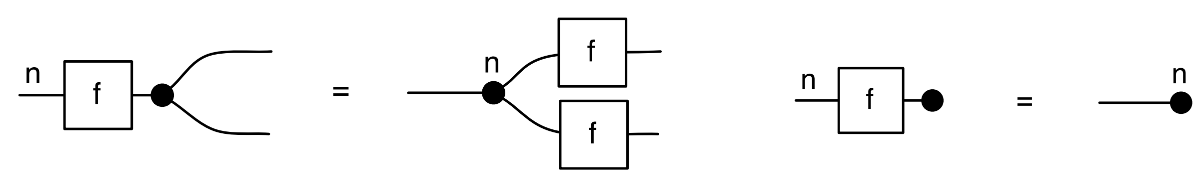
Partial Frobenius algebra, the unit is missing!

Relational theories

(Bonchi, Pavlovic, S. 2017)

- recipe for Frobenius-theories-as-locally-ordered-props
  - add a Frobenius bimonoid structure where monoid is right adjoint to comonoid
  - make all your generators laxly commute with it
  - add your other equations (which may make use of the Frobenius structure)





e.g. $id_0 \leq$ 

Functorial semantics

- For Lawvere theories
  - models = cartesian functors (to $\mathbf{Set}_\times$)
  - homomorphisms = natural transformations
- For partial theories
  - models = cartesian restriction functors (to $\mathbf{Par}_\times$)
  - homomorphisms = lax natural transformations
- For relational theories
  - models = morphisms of cartesian bicategories (to $\mathbf{Rel}_\times$)
  - homomorphisms = lax natural transformations

varieties = locally finitely presentable categories

varieties = definable categories

See Chad Nester's thesis sometime in 2023!

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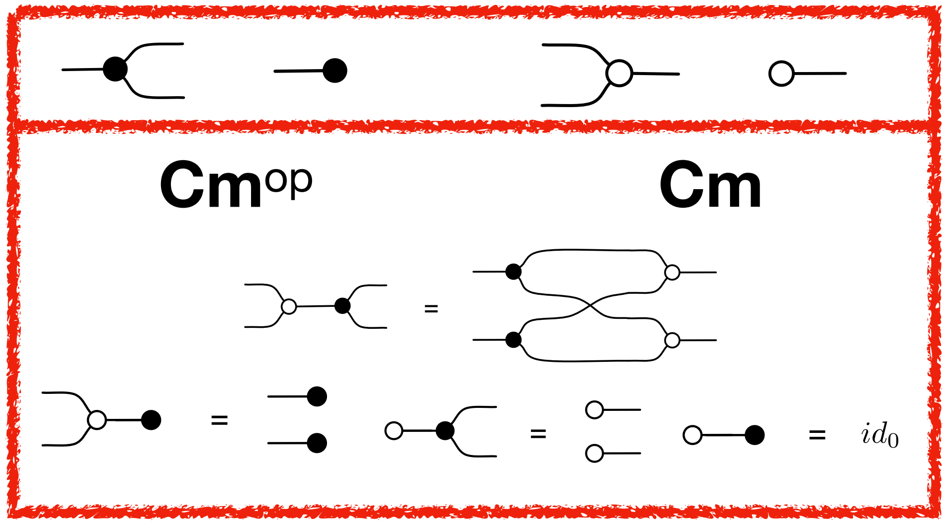
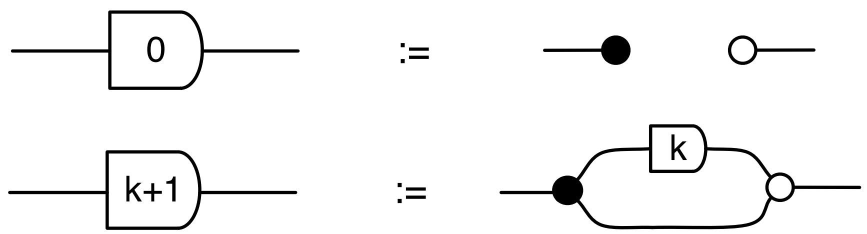
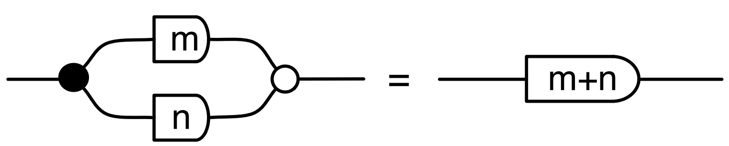
Lawvere theory of commutative monoids = matrices of natural numbers $Mat_{\mathbb{N}}$



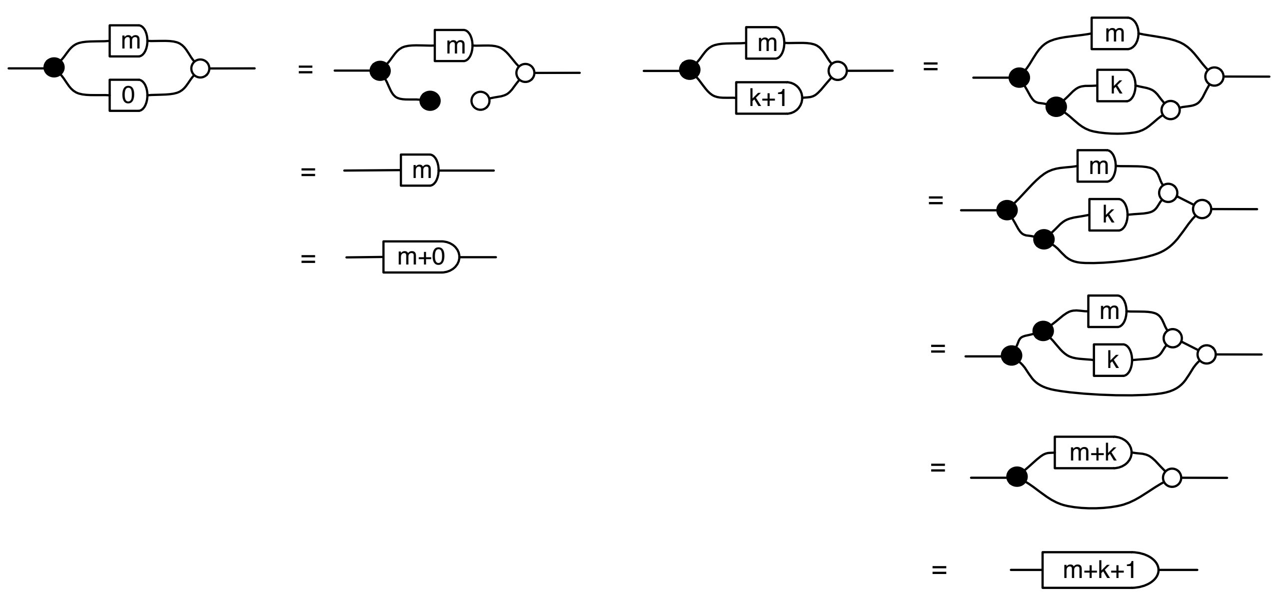
Sugar:



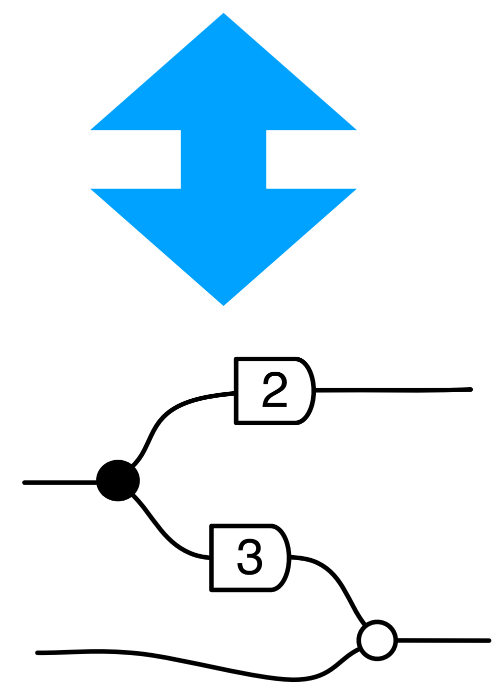
Lemma



Proof



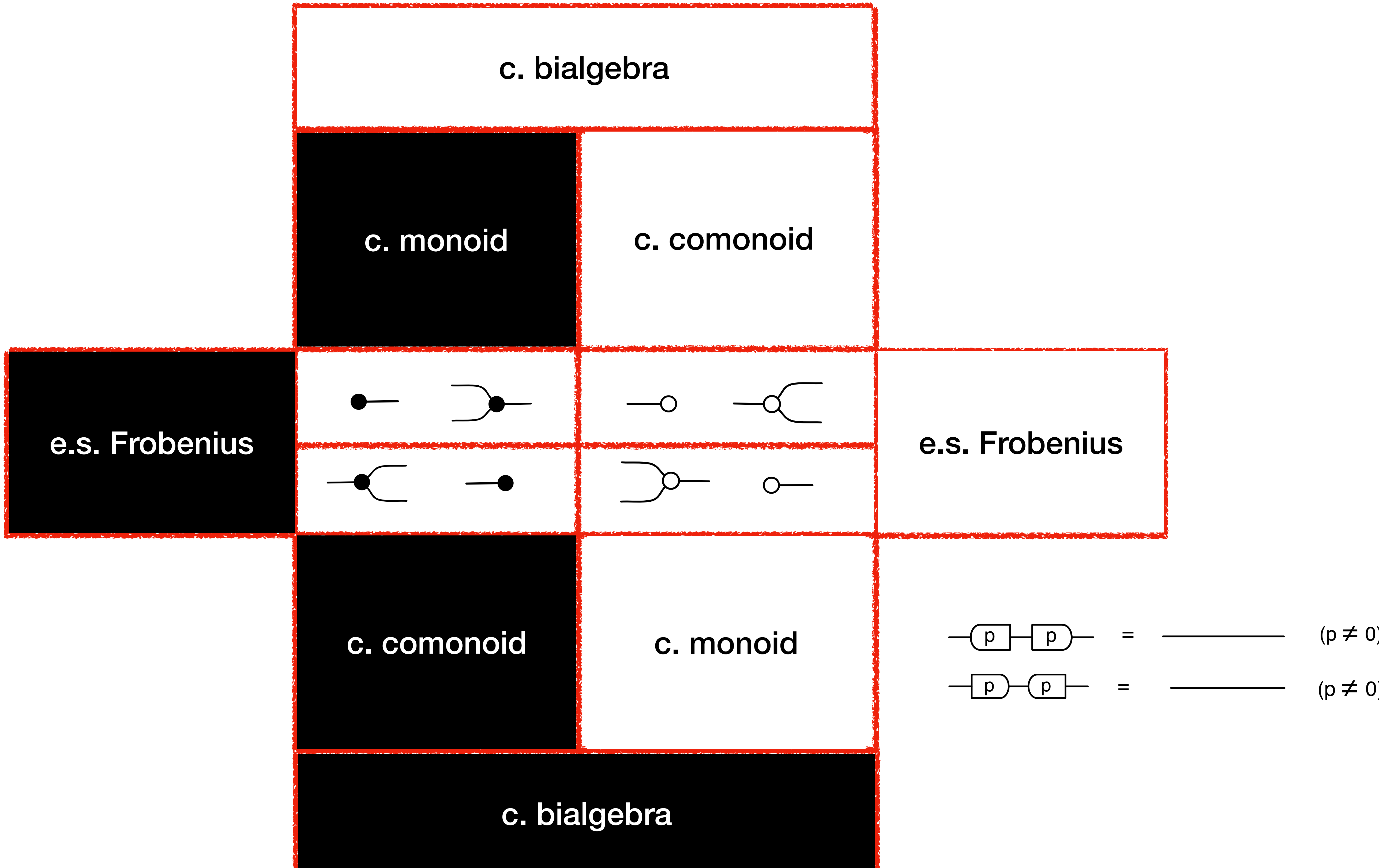
$$\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$



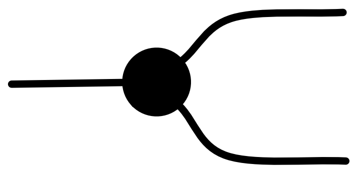
Relational theory of linear relations

- Give a vector space k, $\mathbf{LinRel}_k$ is the smc where
  - objects are natural numbers
  - arrows m to n are relations $R \subseteq k^{m+n}$ that are also k-linear subspaces
- Graphical linear algebra = a presentation of the relational theory of linear relations
- The free model is isomorphic to the symmetric monoidal category $\mathbf{LinRel}_{\mathbf{Q}}$

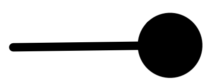
GLA: a presentation of $\mathbf{LinRel}_{\mathbf{Q}}$



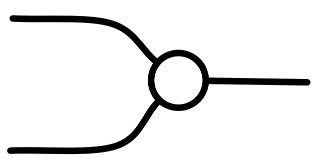
Where do the generators go?



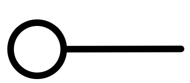
$$\left\{ \left(x, \binom{x}{x}\right) \right\}$$



$$\left\{ \left(x, \bullet\right) \right\}$$

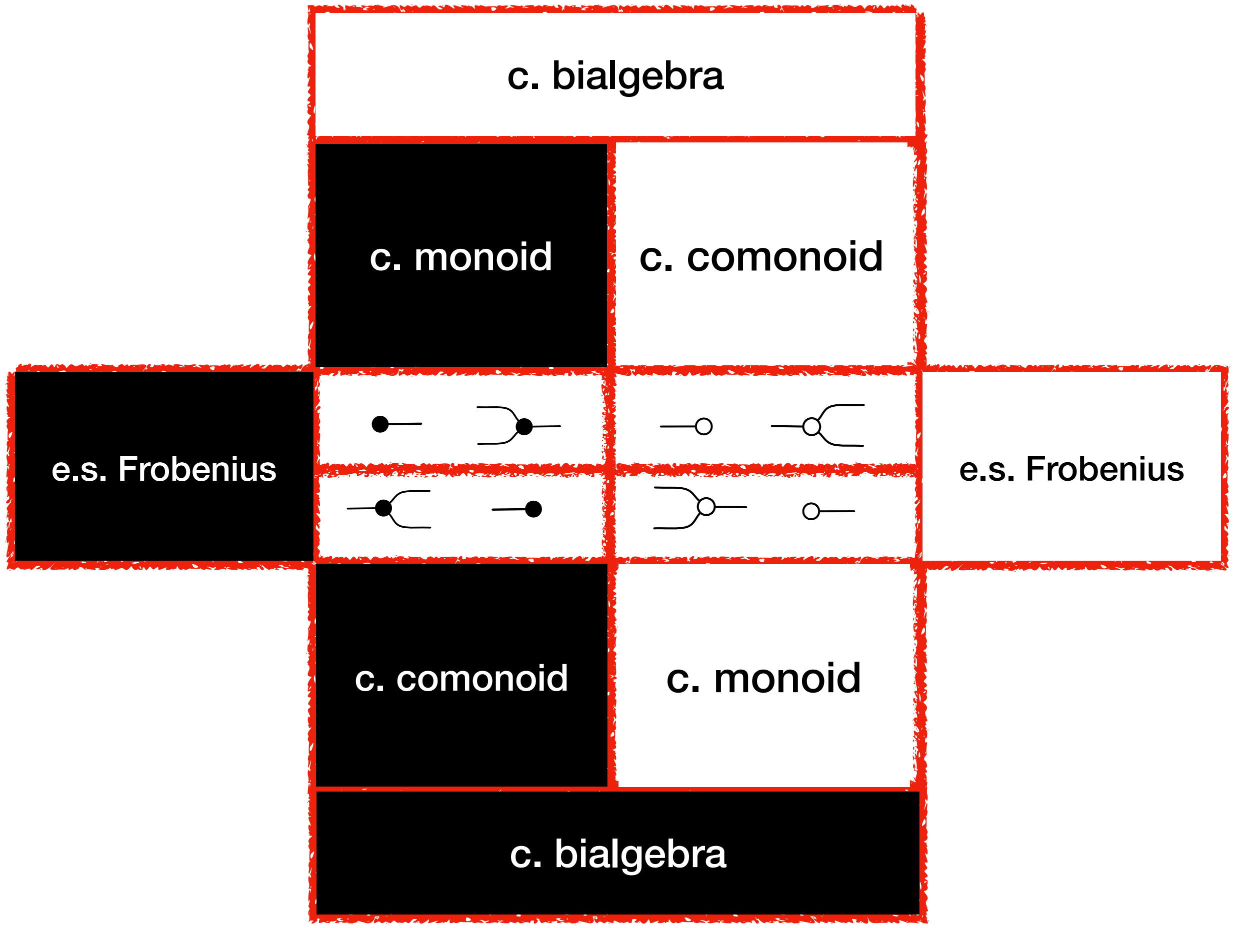


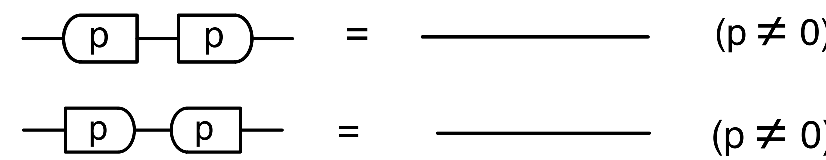
$$\left\{ \left(\binom{x}{y}, x+y\right) \right\}$$



$$\left\{ \left(\bullet, 0\right) \right\}$$

Linear algebra = how these four relations and their opposites interact

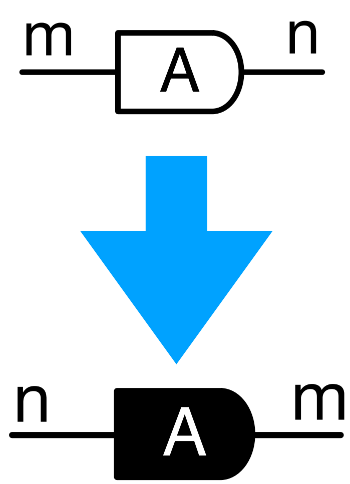




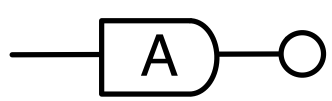
- **Colour**
  - black and white satisfy **exactly the same** equations in the equational theory
  - so every proof is in fact a proof of two theorems: invert the colours!
- **Left-Right**
  - every fact is still a fact when viewed in the mirror

Basic concepts, diagrammatically

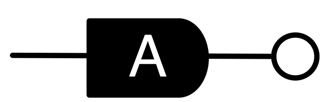
- transpose
  - combine colour and mirror image symmetries



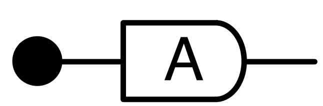
- kernel (nullspace)



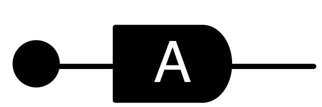
- cokernel (left nullspace)



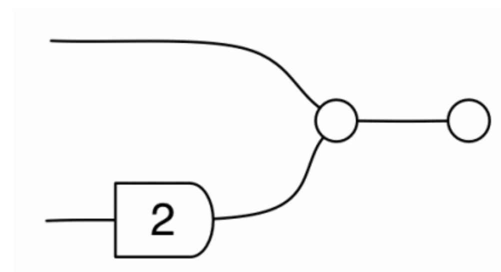
- image (columnspace)



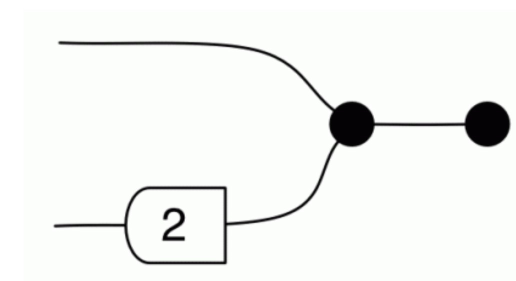
- coimage (rowspace)



Fact. Given a linear subspace R:0->k in **LinRel**, its orthogonal complement R$^\perp$ is its colour inverted diagram



$$\begin{pmatrix} x \\ y \end{pmatrix} \mid x + 2y = 0$$



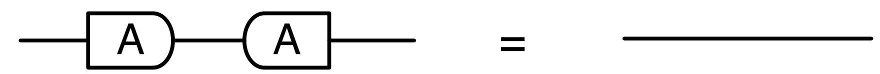
$$\begin{pmatrix} x \\ 2x \end{pmatrix}$$

Corollary. The “fundamental theorem of linear algera”

$$\ker A = \mathrm{im}(A^T)^\perp$$
$$\ker A^T = \mathrm{im}(A)^\perp$$

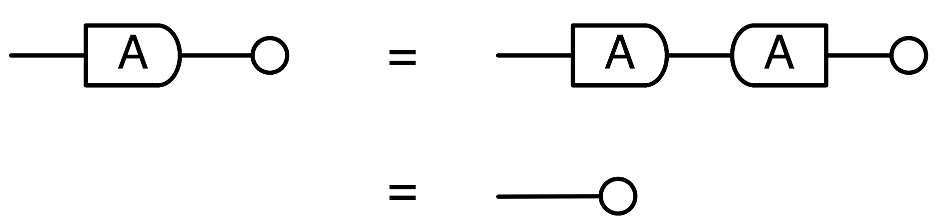
Diagrammatic reasoning in action

Fact. A is injective iff

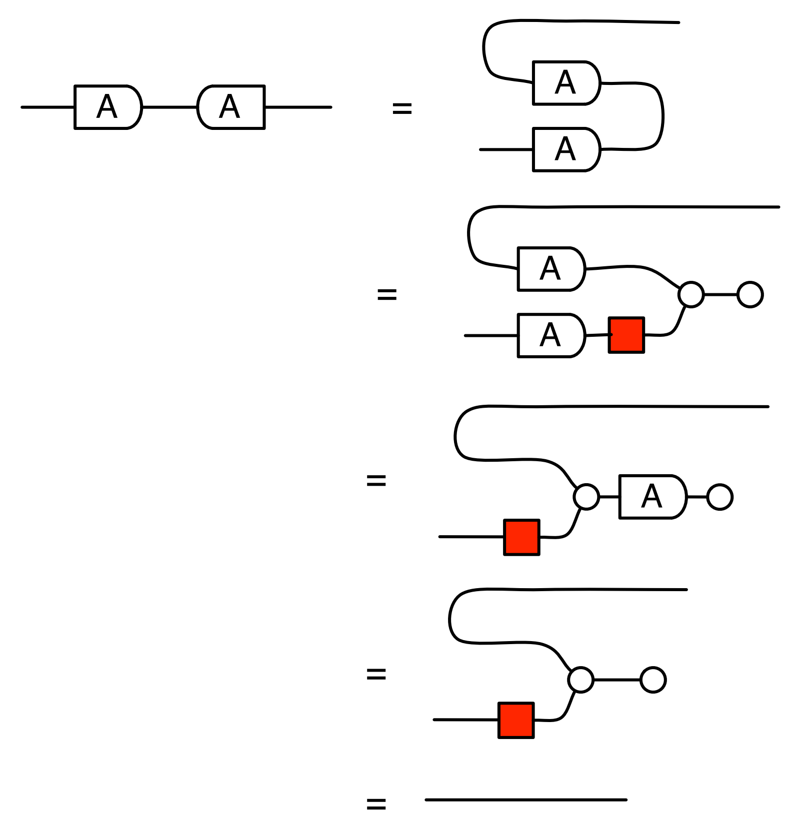


Theorem. A is injective iff ker A = 0

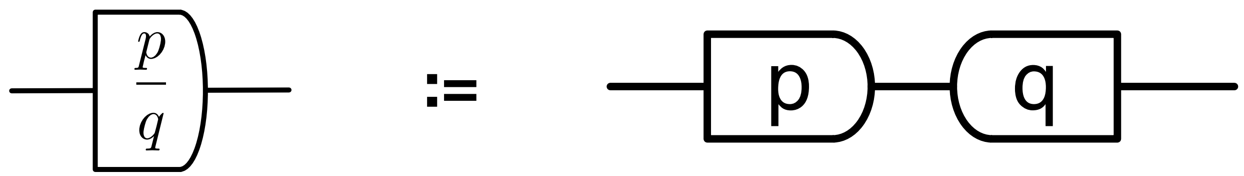
⇒

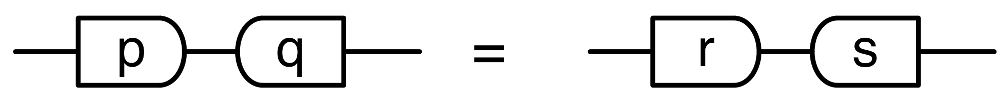


⇐



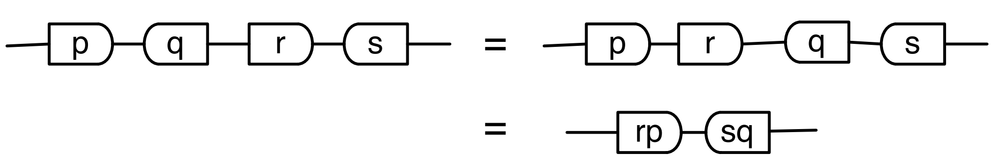
Fun Stuff - Rediscovering Fraction Arithmetic

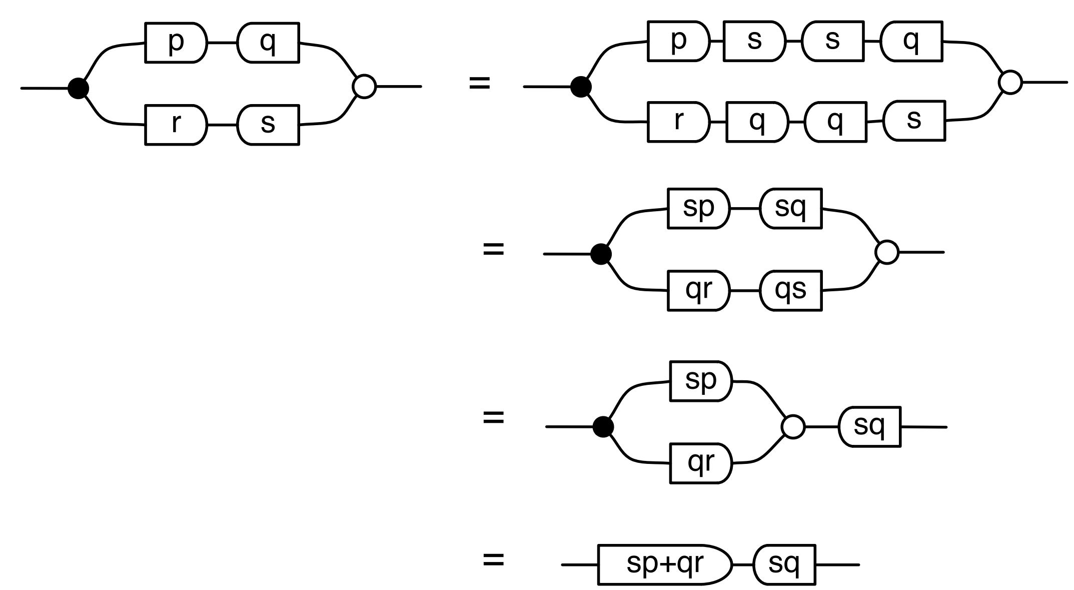




⇔

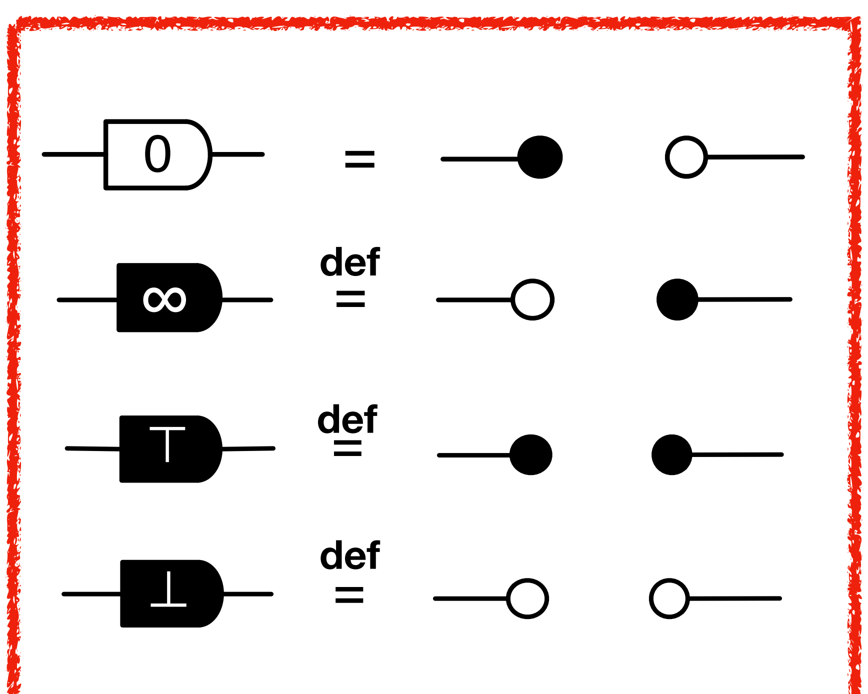
sp = qr





Fun Stuff - Dividing by Zero

- $\mathrm{LinRel}_{\mathbf{Q}}[1,1]$
- projective arithmetic with two additional elements
  - the unique 0-dimensional subspace $\bot = \{ (0,0) \}$
  - The unique 2-dimensional subspace $\top = \{ (x,y) \mid x,y \in \mathbf{Q} \}$



+	0	r/s	∞	⊤	⊥
0	0	r/s	∞	⊤	⊥
p/q	–	(sp+qr)/qs	∞	⊤	⊥
∞	–	–	∞	∞	∞
⊤	–	–	–	⊤	∞
⊥	–	–	–	–	⊥

×	0	r/s	∞	⊤	⊥
0	0	0	⊥	0	⊥
p/q	0	pr/qs	∞	⊤	⊥
∞	⊤	∞	∞	⊤	∞
⊤	⊤	⊤	∞	⊤	∞
⊥	0	⊥	⊥	0	⊥

Plan

- String diagrams
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- **Graphical affine algebra and electrical circuits**

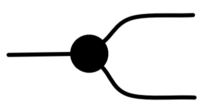
Graphical Affine Algebra

(Bonchi, Piedeleu, S., Zanasi 2019)

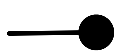
Definition. Given a field **k**, a **k**-affine relation k l is a set $R \subseteq \mathbf{k}^k \times \mathbf{k}^l$ which is either empty, or s.t. there is a k-linear relation C and a vector (**a**,**b**) s.t. R = (**a**,**b**) + C

- *Proposition*: affine relations are closed under composition
- $AffRel_{\mathbf{k}}$ = sub prop of $Rel_{\mathbf{k}}$ where arrows are affine relations

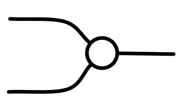
Diagrammatic syntax for k-affine relations



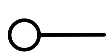
$$\left\{ \left(x, \begin{pmatrix}x\\x\end{pmatrix}\right) \right\}$$



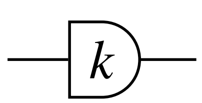
$$\{ (x, \bullet) \}$$



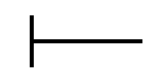
$$\left\{ \left(\begin{pmatrix}x\\y\end{pmatrix}, x+y\right) \right\}$$



$$\{ (\bullet, 0) \}$$



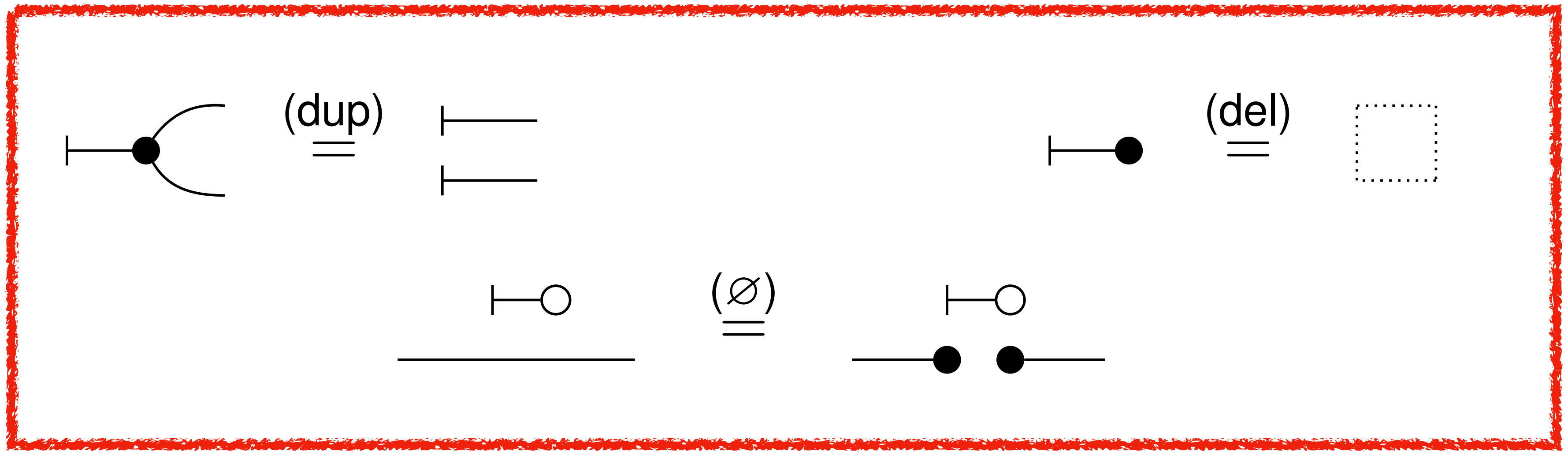
$$\{ (x, kx) \}$$



$$\{ (\bullet, 1) \}$$

Equational characterisation

GAA = **GLA** +



Theorem. **GAA** ≅ $\mathsf{AffRel}_{\mathbf{k}}$

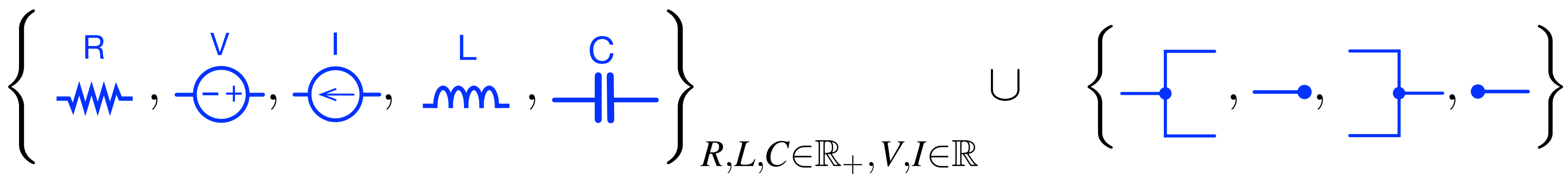
Case study: non passive electrical circuits

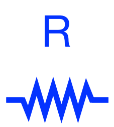
(Boisseau, S. 2021)

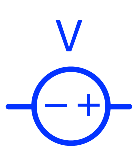
- work with the diagrammatic language for $\mathsf{AffRel}_{\mathbf{R}[x]}$
- introduce a syntactic prop of electrical circuits
- develop diagrammatic reasoning techniques
  - the impedance calculus
- prove classical “theorems” of electrical circuit theory

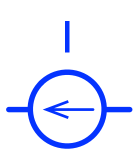
The prop of electrical circuits

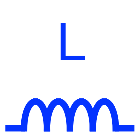
- **ECirc**, free on the following signature

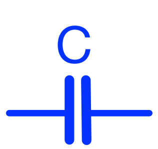


- 

resistor
- 

voltage source
- 

current source
- 

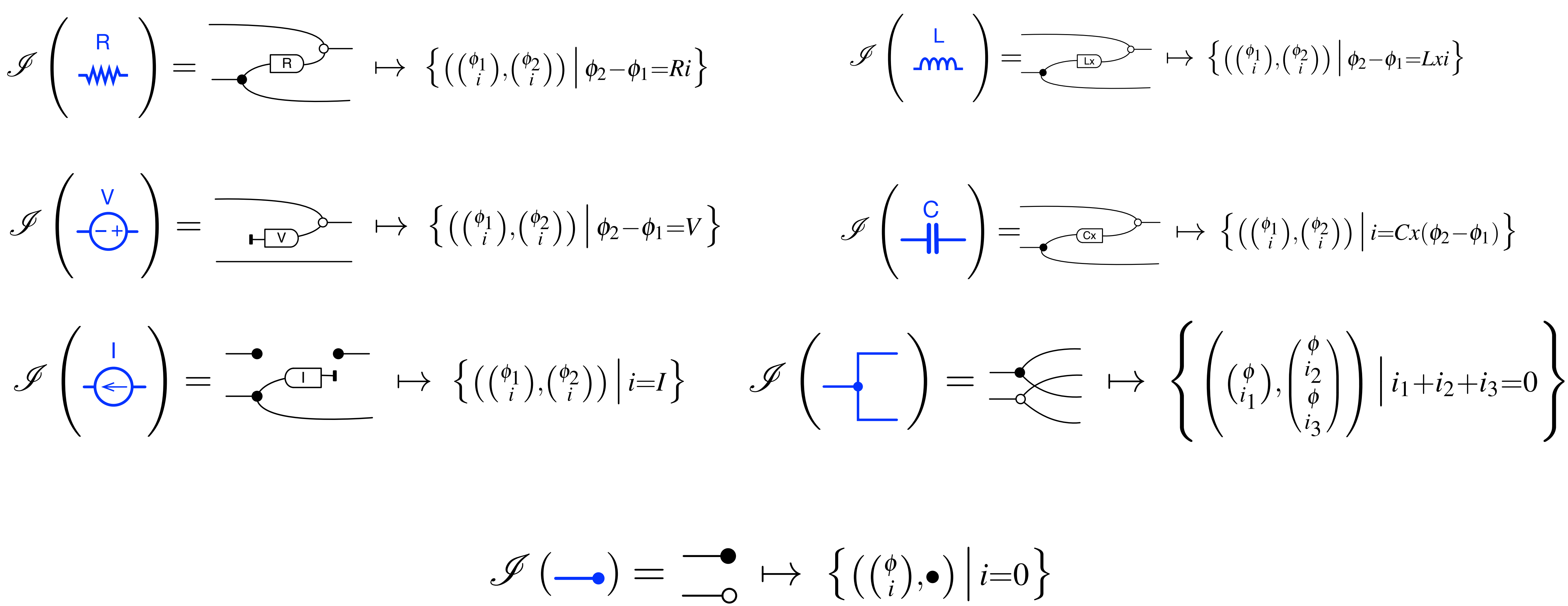
inductor
- 

capacitor

Circuits as GAA diagrams over R[x]

$$\mathscr{I} : \mathsf{ECirc} \to \mathsf{GAA}$$

$$\mathscr{I}(1) = 2$$



$$\mathscr{I}(R) = \ldots \mapsto \left\{\left(\binom{\phi_1}{i},\binom{\phi_2}{i}\right) \,\middle|\, \phi_2-\phi_1=Ri\right\}$$

$$\mathscr{I}(L) = \ldots \mapsto \left\{\left(\binom{\phi_1}{i},\binom{\phi_2}{i}\right) \,\middle|\, \phi_2-\phi_1=Lxi\right\}$$

$$\mathscr{I}(V) = \ldots \mapsto \left\{\left(\binom{\phi_1}{i},\binom{\phi_2}{i}\right) \,\middle|\, \phi_2-\phi_1=V\right\}$$

$$\mathscr{I}(C) = \ldots \mapsto \left\{\left(\binom{\phi_1}{i},\binom{\phi_2}{i}\right) \,\middle|\, i=Cx(\phi_2-\phi_1)\right\}$$

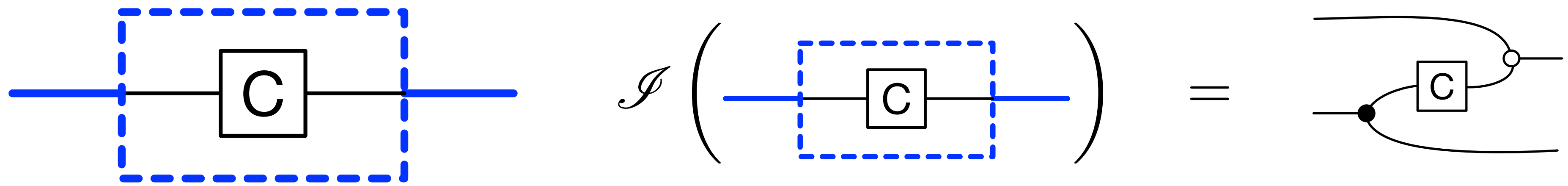
$$\mathscr{I}(I) = \ldots \mapsto \left\{\left(\binom{\phi_1}{i},\binom{\phi_2}{i}\right) \,\middle|\, i=I\right\}$$

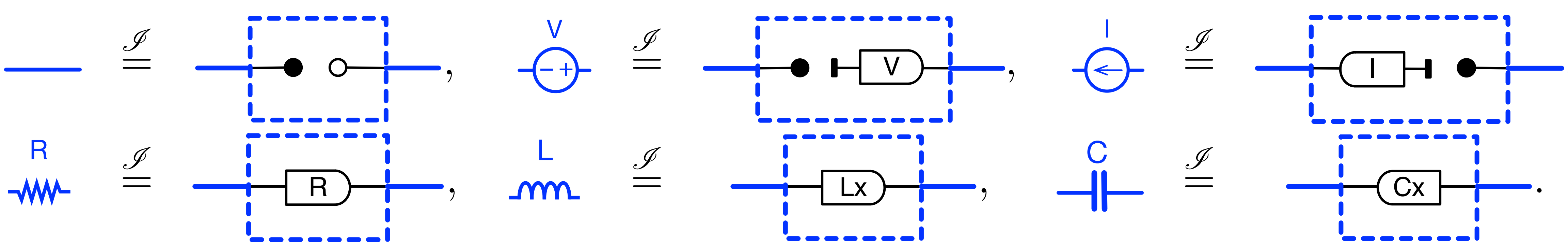
$$\mathscr{I}(\ldots) = \ldots \mapsto \left\{\left(\binom{\phi}{i_1},\begin{pmatrix}\phi\\ i_2\\ \phi\\ i_3\end{pmatrix}\right) \,\middle|\, i_1+i_2+i_3=0\right\}$$

$$\mathscr{I}(\ldots) = \ldots \mapsto \left\{\left(\binom{\phi}{i},\bullet\right) \,\middle|\, i=0\right\}$$

Impedance calculus

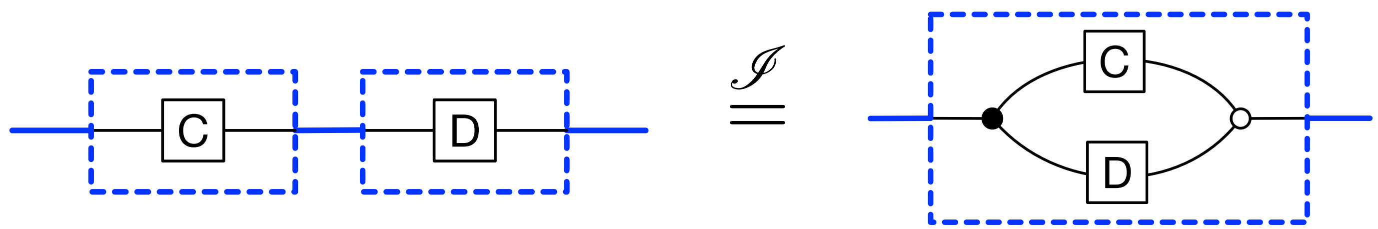
- Extend the signature of **ECirc** with impedance boxes



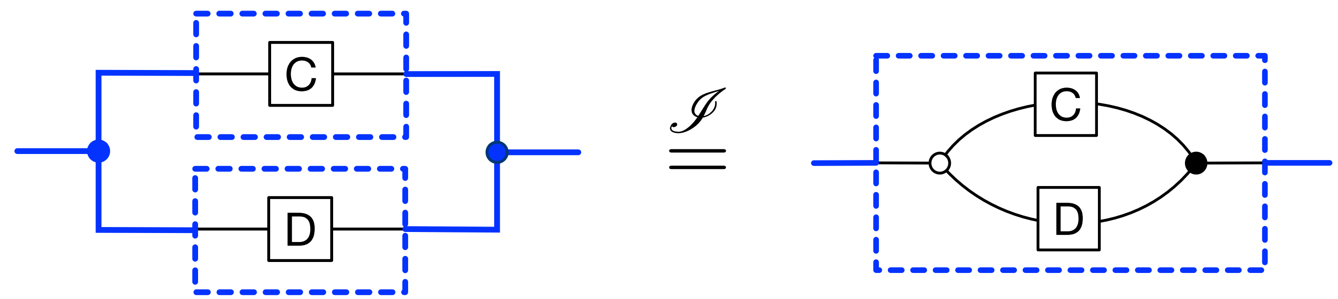


Lemma

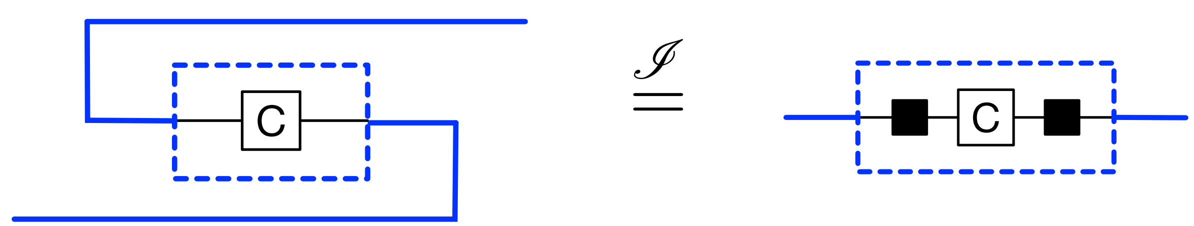
(i)



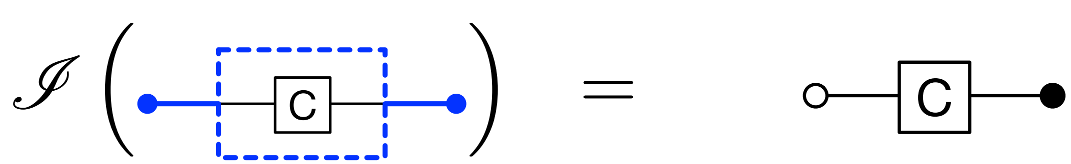
(ii)



(iii)

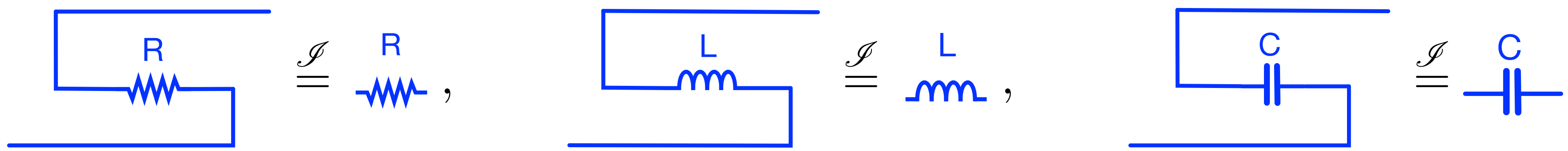


(iv)



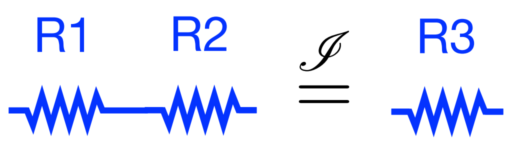
Corollary

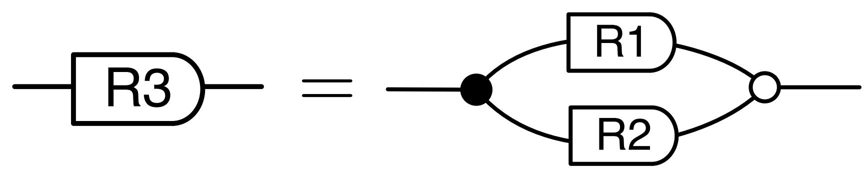
(i) Resistors, inductors and capacitors are “directionless”:

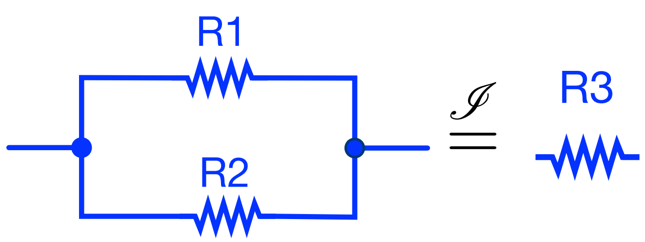


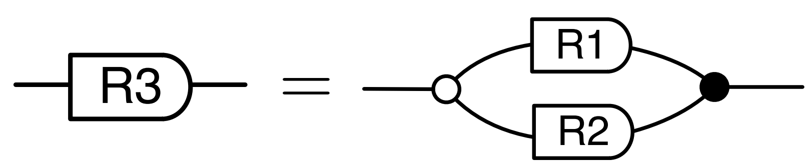
(ii) Reversing the direction of voltage and current sources flips polarities:

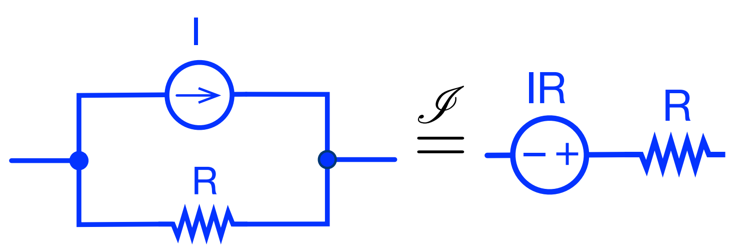
Proposition

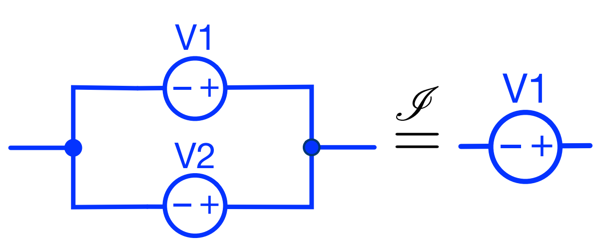
(i) 

where 

(ii) 

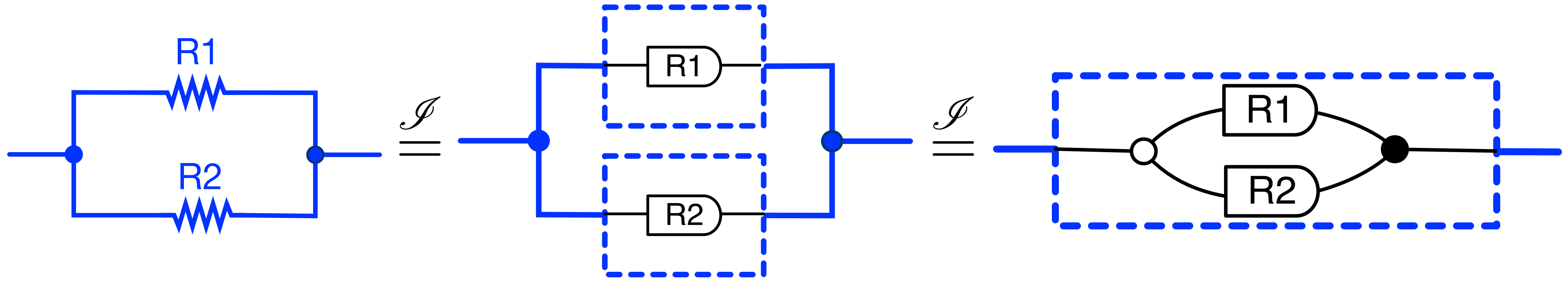
where 

(iii) 

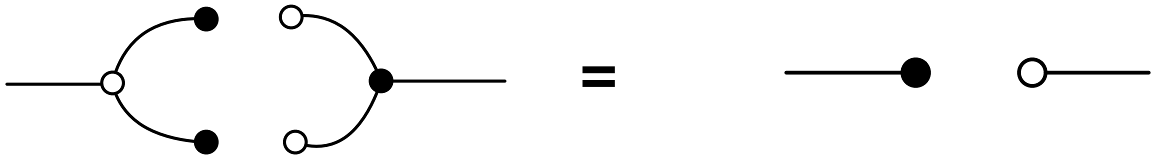
(iv) 

if $V1 = V2$, otherwise its semantics is $\emptyset$ (the empty relation)

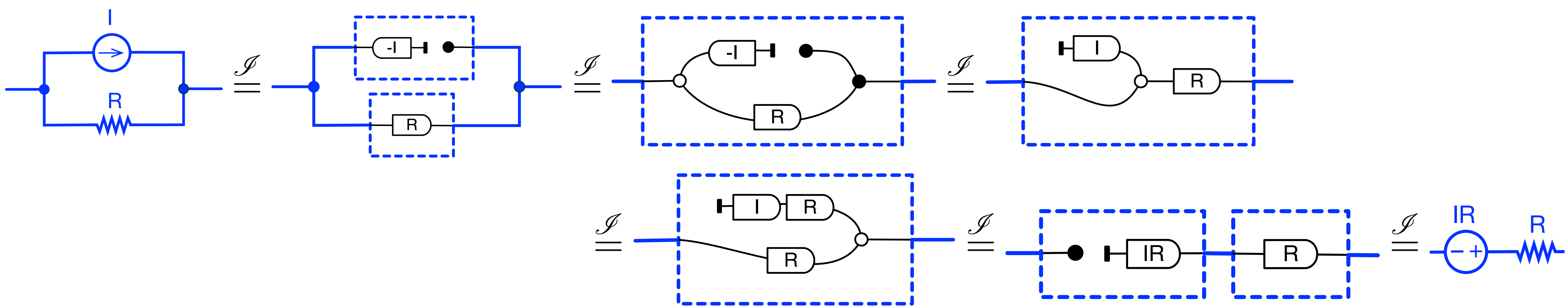
Proof of (ii)



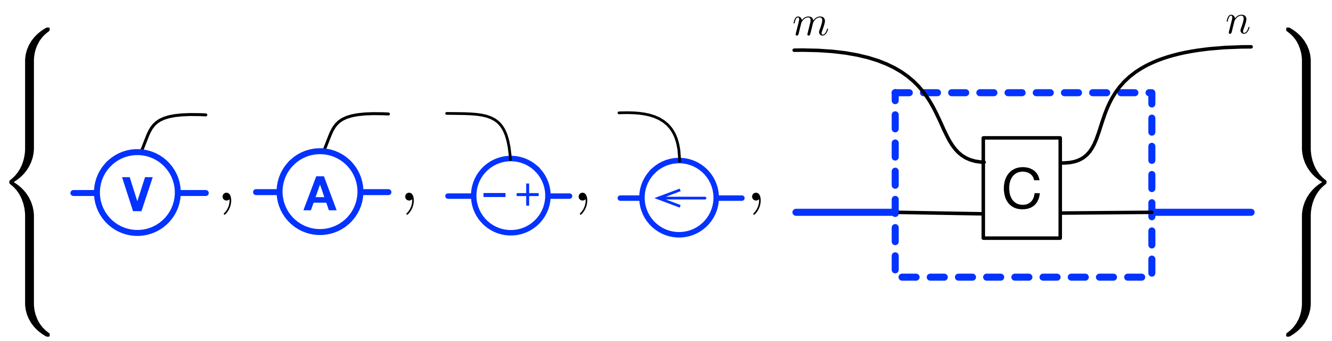
What if R1=R2=0?



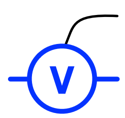
Proof of (iii)

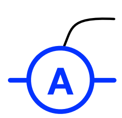


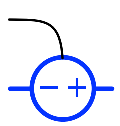
Measuring closed circuits

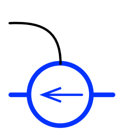


$m,n \in \mathbb{N}, c: m+1 \to n+1$ in GAA

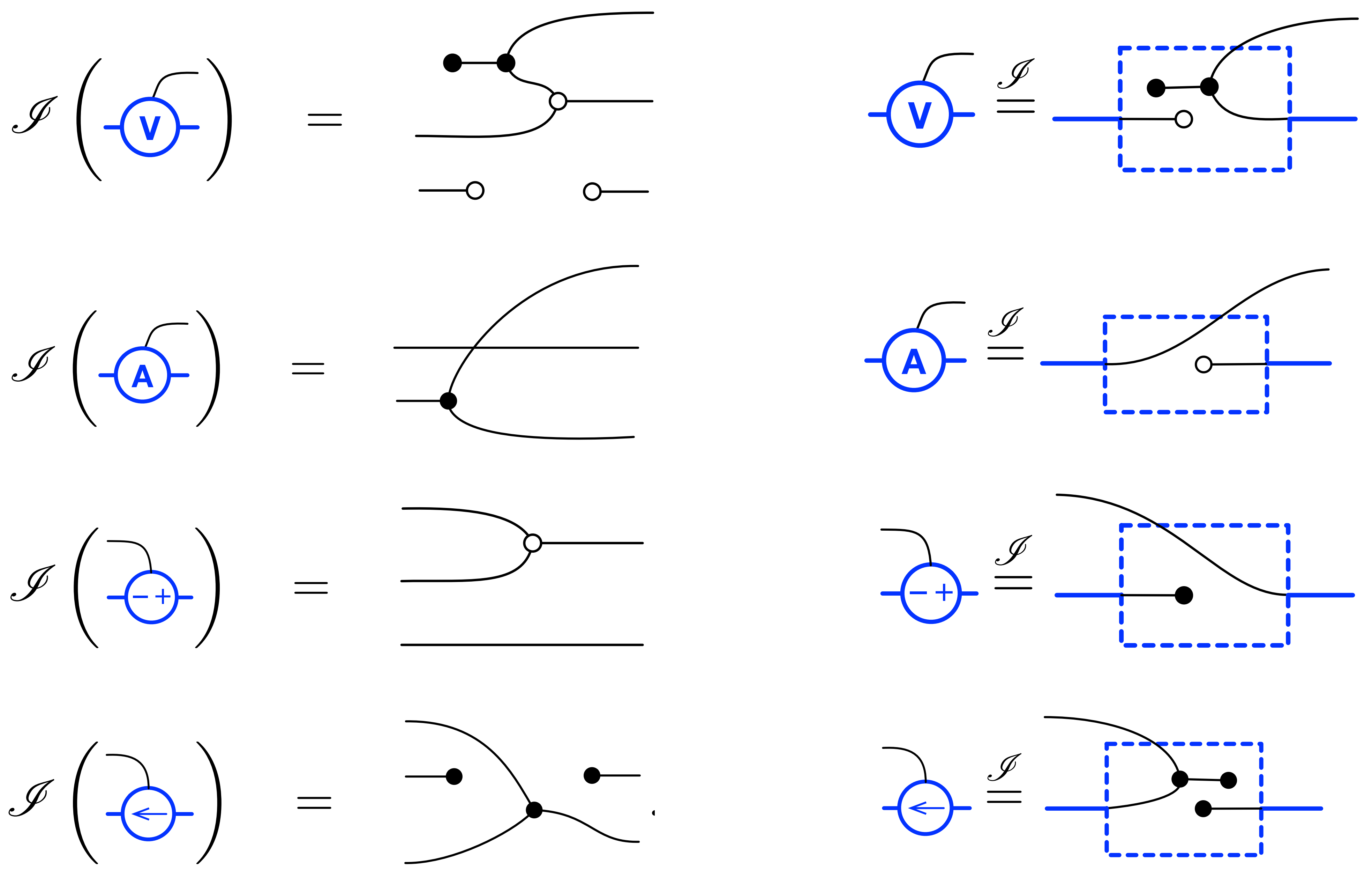
- 

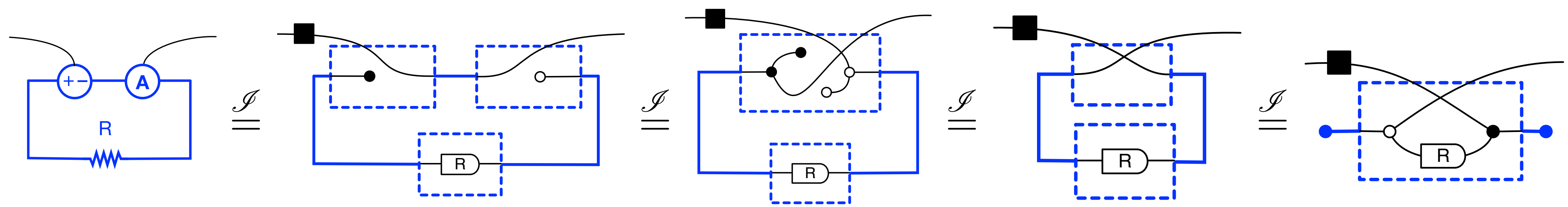
: (•, ••) voltmeter
- 

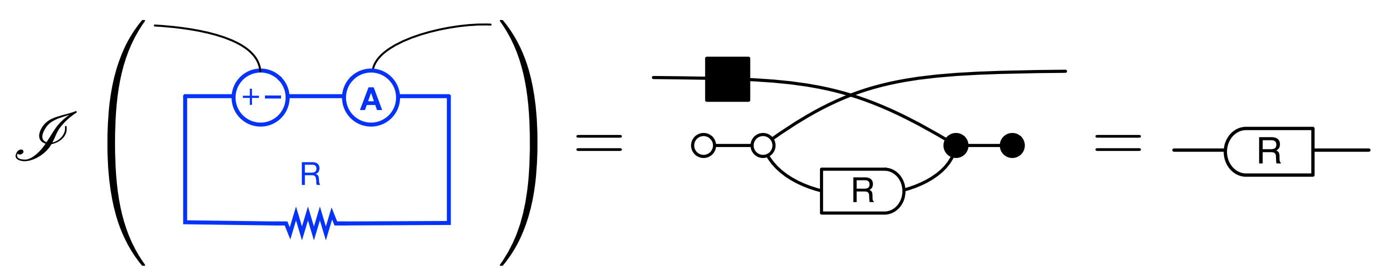
: (•, ••) ammeter
- 

: (••, •) controlled voltage source
-  : (••, •) controlled current source

Compilation to GAA

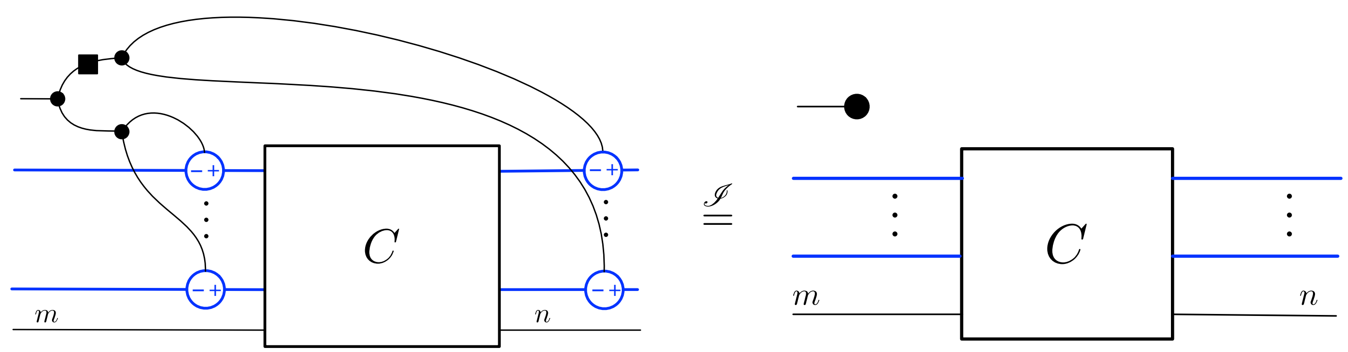



+ −
A
R
R
R
R
R

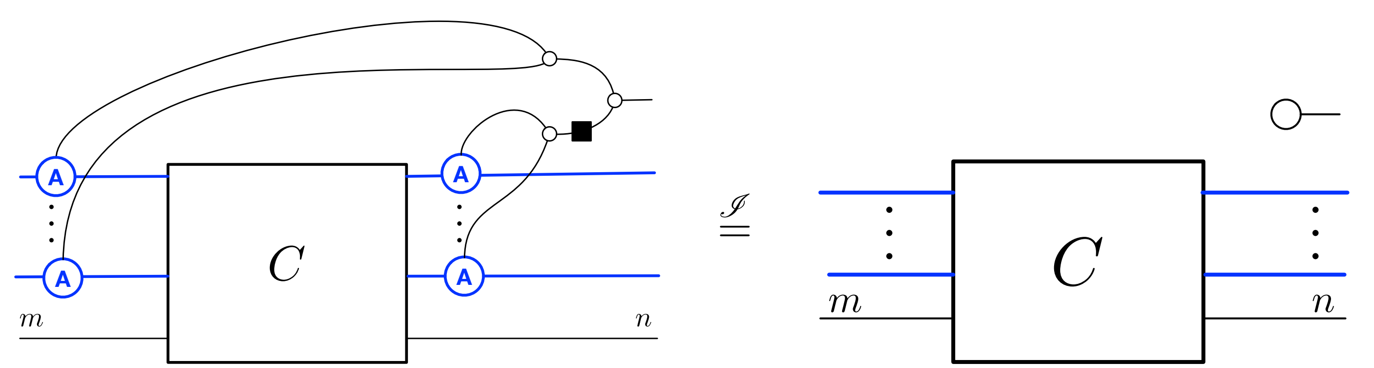

+ −
A
R
R
R

Theorems 1

- **Relativity of potentials.** Adding the same voltage difference to open wires does not change behaviour.

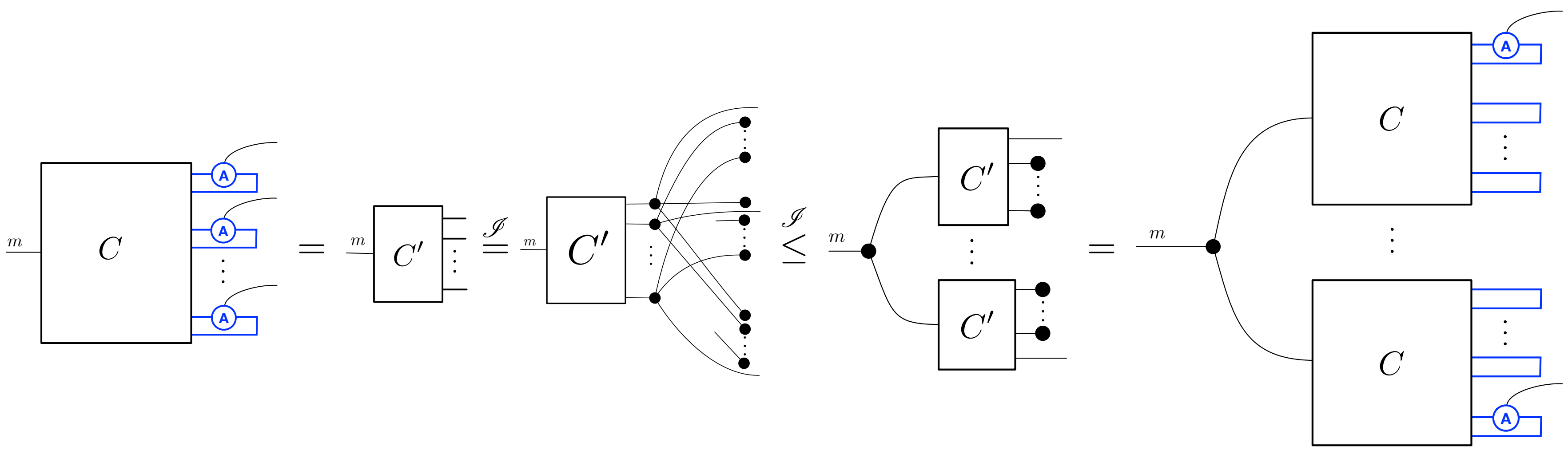


- **Conservation of current.** The sum of incoming current is equal to the sum of outgoing current.

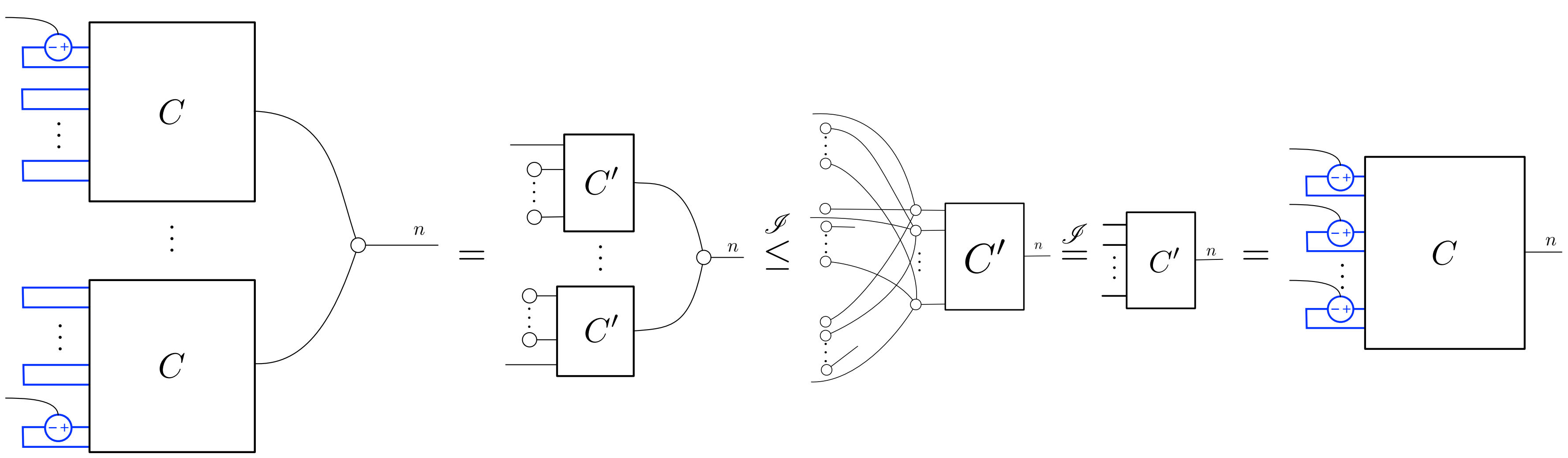


Theorems 2

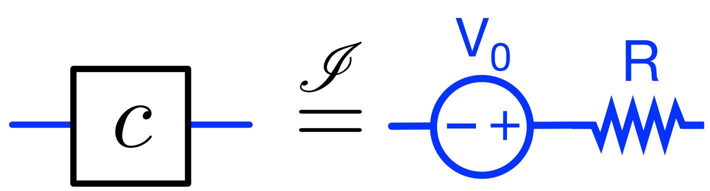
- **Independent measurement theorem.**

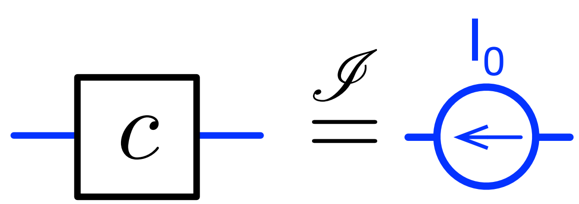


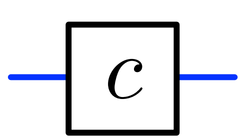
- **Superposition theorem.**



Thévenin’s theorem

- If C is a one port circuit of resistors and independent sources then one of the following is true
  - - 

  - - 

  - -  denotes the empty relation

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