Implicative algebras:
A new foundation for realizability and forcing

Alexandre Miquel

UNIVERSIDAD DE LA REPUBLICA URUGUAY
FACULTAD DE INGENIERIA
EQUIPO DE LÓGICA · UDELAR

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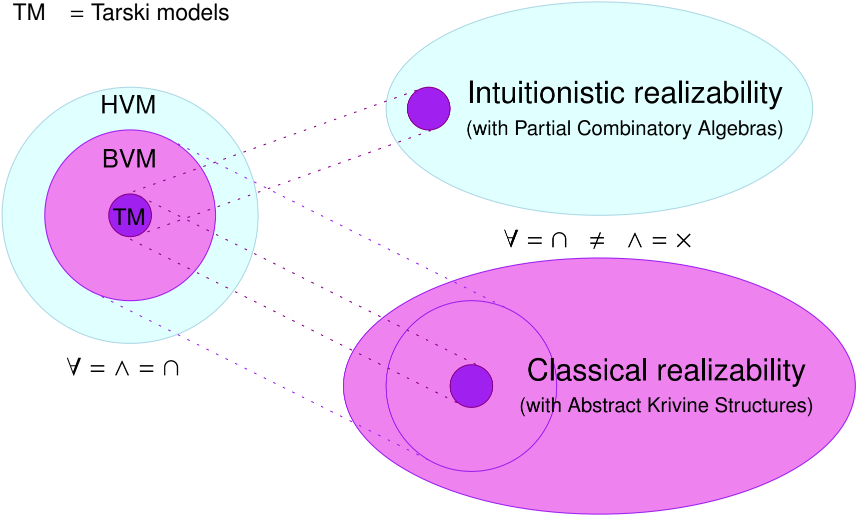
Different notions of models (1/2)

- **Tarski models:** $[\![\phi]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda)$ [Kleene '45]
  - Interprets intuitionistic proofs
  - Independence results in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![\phi]\!] \in \mathfrak{P}(C)$ [Cohen '63]
  - Independence results, in classical theories
    (Negation of continuum hypothesis, Solovay's axiom, etc.)
- **Boolean-valued models:** $[\![\phi]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]
- **Classical realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda_c)$ [Krivine '94, '01, '03, '09–]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing!

Different notions of models (2/2)

HVM = Heyting-valued models $\approx$ Kripke forcing
BVM = Boolean-valued models $\approx$ Cohen forcing
TM = Tarski models

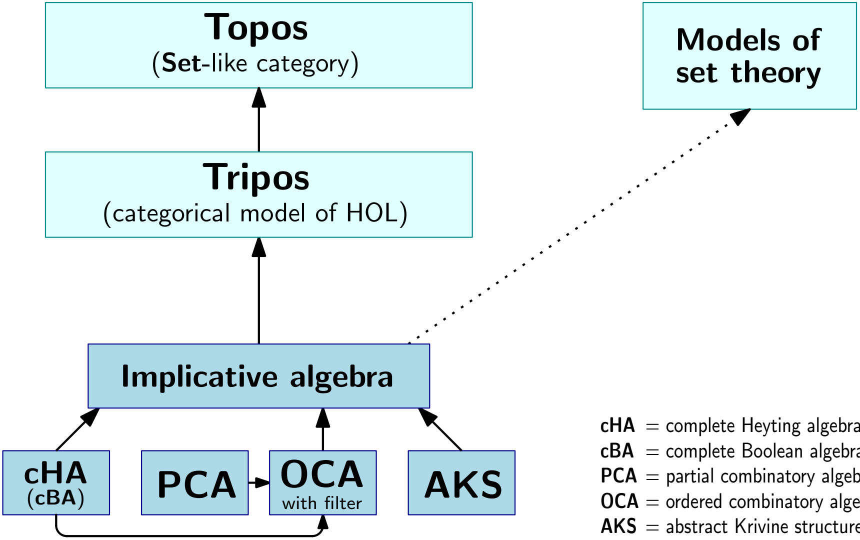


The categorical tradition of realizability

- **Categorical logic** [Lawvere, Tierney '70]
  - Hyperdoctrines = models of 1st order theories
    (Slogan: ∃/∀ are left/right adjoints!)
  - Modern definition of the notion of topos
    (generalizes Grothendieck's definition)
- **Categorical realizability** [Hyland, Johnstone, Pitts '80]
  - Major input from forcing and Boolean-valued models [Scott]
  - Effective topos [Hyland]
  - Notion of tripos and tripos-to-topos construction [Pitts]
  - Generalization to partial combinatory algebras (PCAs)

    ... but incompatible with classical logic
- **What about classical realizability?**

The categorical problem



cHA = complete Heyting algebra
cBA = complete Boolean algebra
PCA = partial combinatory algebra
OCA = ordered combinatory algebra
AKS = abstract Krivine structure

Unifying all kinds of models

Aim: Define an algebraic structure to encompass:

- Complete Heyting Algebras (for Heyting-valued models, Kripke forcing)
- Complete Boolean Algebras (for Boolean-valued models, Cohen forcing)
- Partial Combinatory Algebras (for Intuitionistic realizability)
- Ordered Combinatory Algebras (for Intuitionistic realizability)
- Abstract Krivine Structures (for Classical realizability)

Implicative algebras can be used to construct:

- Categorical models (triposes, toposes)
- Models of (intuitionistic/classical) set theory

Underlying ideas are reminiscent from earlier work of

- Ruyer '07, Streicher '13 (and many others!)

Plan

1. Introduction
2. Implicative structures
3. Separation
4. The implicative tripos
5. Conclusion

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Implicative structures

Definition (Implicative structure)

An implicative structure is a triple $(\mathscr{A}, \preccurlyeq, \to)$ where

(1) $(\mathscr{A}, \preccurlyeq)$ is a complete (meet semi-)lattice

(2) $(\to) : \mathscr{A}^2 \to \mathscr{A}$ is a binary operation such that:

(2a) $a' \preccurlyeq a, \;\; b \preccurlyeq b'$ entails $(a \to b) \preccurlyeq (a' \to b')$ $\qquad (a, a', b, b' \in \mathscr{A})$

(2b) $\bigcurlywedge_{b \in B} (a \to b) \; = \; a \to \bigcurlywedge_{b \in B} b$ $\qquad (a \in \mathscr{A},\, B \subseteq \mathscr{A})$

- Write $\bot$ (resp. $\top$) the smallest (resp. largest) element of $\mathscr{A}$
- When $B = \varnothing$, axiom (2b) gives: $(a \to \top) = \top$ $\qquad (a \in \mathscr{A})$

Examples of implicative structures

- Complete Heyting algebras $(\mathscr{A}, \preccurlyeq)$, where $\to$ is defined by:

$$a \to b \;:=\; \max\{c \in \mathscr{A} \;:\; (c \curlywedge a) \preccurlyeq b\}$$ (Heyting's implication)

$+$ complete Boolean algebras (as a particular case of Heyting algebras)

- Given a total combinatory algebra $(P, \cdot, \mathtt{k}, \mathtt{s})$, we let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(P)$
  - $a \preccurlyeq b \;:=\; a \subseteq b$
  - $a \to b \;:=\; \{z \in P : \forall x \in a,\; z \cdot x \in b\}$ (Kleene's implication)

Note: if we do the same with a partial combinatory algebra, we only get a quasi-implicative structure, where $(a \to \top) \neq \top$

$+$ similar construction for ordered combinatory algebras (OCA)

- Given an abstract Krivine structure $(\mathbf{\Lambda}, \mathbf{\Pi}, \ldots, \mathrm{PL}, \perp\!\!\!\perp)$, we let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(\mathbf{\Pi})$
  - $a \preccurlyeq b \;:=\; a \supseteq b$
  - $a \to b \;:=\; a^{\perp\!\!\!\perp} \cdot b$ (Krivine's implication)

Viewing truth values as generalized realizers: a manifesto

1. Elements of an implicative structure are primarily intended to represent truth values. But since λ-abstraction and application are definable in such a structure (cf next slide), we can see:
   - each realizer as a particular truth value;
   - each truth value as a generalized realizer
2. So that we get the ultimate Curry-Howard identification:

   Realizer = **Program** = **Formula** = **Type**

3. In this setting, the relation $a \preccurlyeq b$ may read:
   - a is a subtype of b (viewing a and b as truth values)
   - a has type b (viewing a as a realizer, b as a truth value)
   - a is more defined than b (viewing a and b as realizers)
4. In particular: subtyping ($\preccurlyeq$) = reverse Scott ordering ($\sqsupseteq$)

Encoding application & abstraction

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Application & Abstraction)

Given $a, b \in \mathscr{A}$ and a function $f : \mathscr{A} \to \mathscr{A}$, we let:

$$ab \;:=\; \bigwedge\{c \in \mathscr{A} \;:\; a \preccurlyeq (b \to c)\} \qquad \text{(application)}$$

$$\boldsymbol{\lambda} f \;:=\; \bigwedge_{a \in \mathscr{A}} (a \to f(a)) \qquad \text{(abstraction)}$$

- **Properties:**
  1. If $a \preccurlyeq a'$ and $b \preccurlyeq b'$, then $ab \preccurlyeq a'b'$ (Monotonicity)
  2. If $f \preccurlyeq g$ (pointwise), then $\boldsymbol{\lambda} f \preccurlyeq \boldsymbol{\lambda} g$ (Monotonicity)
  3. $(\boldsymbol{\lambda} f)a \;\preccurlyeq\; f(a)$ (β-reduction)
  4. $a \;\preccurlyeq\; \boldsymbol{\lambda}(x \mapsto ax)$ (η-expansion)
  5. $ab \preccurlyeq c$ iff $a \preccurlyeq (b \to c)$ (Adjunction)

Encoding the λ-calculus

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

- To each closed λ-term t with parameters (i.e. constants) in $\mathscr{A}$, we associate a truth value $t^{\mathscr{A}} \in \mathscr{A}$:

$$\begin{array}{rcl} a^{\mathscr{A}} & := & a \\ (\lambda x\,.\,t)^{\mathscr{A}} & := & \boldsymbol{\lambda}(a \mapsto (t\{x := a\})^{\mathscr{A}}) \\ (tu)^{\mathscr{A}} & := & t^{\mathscr{A}}\,u^{\mathscr{A}} \end{array}$$

- **Properties:**
  - β-rule: If $t \twoheadrightarrow_{\beta} t'$, then $(t)^{\mathscr{A}} \preccurlyeq (t')^{\mathscr{A}}$
  - η-rule: If $t \twoheadrightarrow_{\eta} t'$, then $(t)^{\mathscr{A}} \succcurlyeq (t')^{\mathscr{A}}$
- **Remarks:**
  - This is *not* a denotational model of the λ-calculus!
  - Map $t \mapsto t^{\mathscr{A}}$ is not injective in general, even on $\beta\eta$-normal forms

Remarkable identities

- In any implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ we have:

$$\mathbf{I}^{\mathscr{A}} := (\lambda x\,.\,x)^{\mathscr{A}} = \bigcurlywedge_{a}(a \to a)$$

$$\mathbf{K}^{\mathscr{A}} := (\lambda xy\,.\,x)^{\mathscr{A}} = \bigcurlywedge_{a,b}(a \to b \to a)$$

$$\mathbf{S}^{\mathscr{A}} := (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} = \bigcurlywedge_{a,b,c}((a \to b \to c) \to (a \to b) \to a \to c)$$

$+$ similar equalities for $\mathbf{C} \equiv \lambda xyz\,.\,xzy$ and $\mathbf{W} \equiv \lambda xy\,.\,xyy$

- By analogy, we let:

$$\text{cc}^{\mathscr{A}} := \bigcurlywedge_{a,b}(((a \to b) \to a) \to a) \qquad \text{(Peirce's law)}$$

From this, we extend the encoding of the λ-calculus to all λ-terms enriched with the constant cc (= proof-like λ_c-terms)

Particular case: $\mathscr{A}$ is a complete Heyting algebra

Complete Heyting/Boolean algebras are the particular implicative structures $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ where $\to$ is defined from $\preccurlyeq$ by

$$a \to b \;:=\; \max\{c \in \mathscr{A} \;:\; (c \curlywedge a) \preccurlyeq b\}$$

Remark: Complete Heyting/Boolean algebras are the structures underlying forcing (in the sense of Kripke or Cohen)

Proposition

When $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra:

1. For all $a, b \in \mathscr{A}$: $ab = a \curlywedge b$ (application = binary meet)
2. For each closed λ-term t: $(t)^{\mathscr{A}} = \top$
3. Moreover, when $\mathscr{A}$ is a Boolean algebra: $\text{œ}^{\mathscr{A}} = \top$

Logical strength of an implicative structure

- **Warning!** We may have $(t)^{\mathscr{A}} = \bot$ for some closed λ-term t.

  Intuitively, this means that the corresponding term is inconsistent in (the logic represented by) the implicative structure $\mathscr{A}$

- We say that the implicative structure $\mathscr{A}$ is:
  - intuitionistically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms
  - classically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms with cc

- **Examples:**
  - Every non-degenerated complete Heyting algebra is int. consistent
  - Every non-degenerated complete Boolean algebra is class. consistent
  - Implicative structures induced by CAs/OCAs are int. consistent
  - Every Krivine realizability structure whose pole $⫫$ is coherent (cf [Krivine'12]) is classically consistent

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Separators (1/3)

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Separator)

A separator of $\mathscr{A}$ is a subset $S \subseteq \mathscr{A}$ such that:

(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$ (upwards closed)

(2) $\mathbf{K}^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \in S$ and $\mathbf{S}^{\mathscr{A}} = (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} \in S$

(3) If $(a \to b) \in S$ and $a \in S$, then $b \in S$ (modus ponens)

We say that S is consistent (resp. classical) when $\bot \notin S$ (resp. $\text{æ}^{\mathscr{A}} \in S$)

Remarks:

- Under (1), axiom (3) is equivalent to:

  $(3')$ If $a, b \in S$, then $ab \in S$ (closure under application)

- In general, separators are not closed under binary meets

Separators (2/3)

- **Intuition:** Separator $S \subseteq \mathscr{A}$ = criterion of truth (in $\mathscr{A}$)
- When $\mathscr{A}$ is a complete Heyting/Boolean algebra, a separator is the same as a filter (since application = binary meet)

  But in general, separators are not filters (not closed under binary meets)

Definition (Intuitionistic and classical cores)

The smallest intuitionistic/classical separators of $\mathscr{A}$ are:

$$S_J^0(\mathscr{A}) := \uparrow\{(t)^{\mathscr{A}} : t \text{ closed } \lambda\text{-term}\} \qquad \text{(intuitionistic core)}$$

$$S_K^0(\mathscr{A}) := \uparrow\{(t)^{\mathscr{A}} : t \text{ closed } \lambda\text{-term with } \text{cc}\} \qquad \text{(classical core)}$$

writing $\uparrow B$ the upwards closure of a subset $B \subseteq \mathscr{A}$

- Note that:
  - When $\mathscr{A}$ is a complete Heyting algebra: $S_J^0(\mathscr{A}) = \{\top\}$
  - When $\mathscr{A}$ is a complete Boolean algebra: $S_K^0(\mathscr{A}) = \{\top\}$

Separators (3/3)

Separators can be used the same way as filters:

- Separators are closed under λ-constructions:

If $a_1, \ldots, a_n \in S$, then $t^{\mathscr{A}}(a_1, \ldots, a_n) \in S$ (for all λ-terms $t(x_1, \ldots, x_n)$)

- We can define the separator generated by an arbitrary subset X:

$$\mathsf{Sep}(X) \ := \ \uparrow\{t^{\mathscr{A}} \ : \ t \text{ closed } \lambda\text{-term with parameters in } X\}$$

  - We have $S_J^0(\mathscr{A}) = \mathsf{Sep}(\varnothing)$ and $S_K^0(\mathscr{A}) = \mathsf{Sep}(\{\text{œ}^{\mathscr{A}}\})$
  - Deduction lemma: $(a \to b) \in \mathsf{Sep}(X)$ iff $b \in \mathsf{Sep}(X \cup \{a\})$

- We can even define ultraseparators as the maximal consistent separators. As for (ultra)filters, we have:

$$S \subset \mathscr{A} \text{ is an ultraseparator} \quad \text{iff} \quad \mathscr{A}/S = \mathbf{2}$$

Beware! Some ultraseparators $S \subset \mathscr{A}$ are non-classical (i.e. $\text{œ}^{\mathscr{A}} \notin S$)

Interpreting first-order logic

- Formulas of first-order logic are interpreted by:

$$
\begin{aligned}
[\![\phi \Rightarrow \psi]\!] &= [\![\phi]\!] \to [\![\psi]\!] \\
[\![\neg\phi]\!] &= [\![\phi]\!] \to \bot \\
[\![\phi \wedge \psi]\!] &= \bigcurlywedge_{a \in \mathscr{A}} \big(([\![\phi]\!] \to [\![\psi]\!] \to a) \to a\big) \\
[\![\phi \vee \psi]\!] &= \bigcurlywedge_{a \in \mathscr{A}} \big(([\![\phi]\!] \to a) \to ([\![\psi]\!] \to a) \to a\big) \\
[\![\forall x\, \phi(x)]\!] &= \bigcurlywedge_{v \in \mathscr{M}} [\![\phi(v)]\!] \\
[\![\exists x\, \phi(x)]\!] &= \bigcurlywedge_{a \in \mathscr{A}} \Big(\bigcurlywedge_{v \in \mathscr{M}} ([\![\phi(v)]\!] \to a) \;\to\; a\Big)
\end{aligned}
$$

(where $\mathscr{M}$ is the domain of the interpretation)

Theorem (Soundness)

If $\vdash_{\mathsf{LJ}} \phi$ (resp. $\vdash_{\mathsf{LK}} \phi$), then $[\![\phi]\!] \in S^0_J(\mathscr{A})$ (resp. $[\![\phi]\!] \in S^0_K(\mathscr{A})$)

Implicative algebras

Definition (Implicative algebra)

An implicative algebra is a quadruple $(\mathscr{A}, \preccurlyeq, \to, S)$ where

- $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
- $S \subseteq \mathscr{A}$ is a separator

- The separator $S \subseteq \mathscr{A}$ induces a preorder of entailment:

$$a \vdash_S b \quad :\equiv \quad (a \to b) \in S \qquad\qquad (\text{for all } a, b \in \mathscr{A})$$

- The poset reflection of $(\mathscr{A}, \vdash_S)$ is written $\mathscr{A}/S$

Proposition

1. The poset $\mathscr{A}/S$ is a Heyting algebra
2. If $œ^{\mathscr{A}} \in S$, then $\mathscr{A}/S$ is a Boolean algebra

Remark: The induced Heyting algebra $\mathscr{A}/S$ is in general not complete

Non deterministic choice and filters

- In the theory of implicative algebras, separators play the same role as filters in the theory of Heyting algebras.

  However, separators $S \subseteq \mathscr{A}$ are in general *not* filters:

$$\begin{array}{ccc} a, b \in S & \Rightarrow & ab \in S \\ a, b \in S & \not\Rightarrow & a \curlywedge b \in S \end{array}$$

- Given an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$, we let:

$$\pitchfork^{\mathscr{A}} \quad := \quad \bigcurlywedge_{a,b \in \mathscr{A}} (a \to b \to a \curlywedge b) \qquad \text{(non deterministic choice)}$$

$$\text{p-or}^{\mathscr{A}} \quad := \quad (\bot \to \top \to \bot) \curlywedge (\top \to \bot \to \bot) \qquad \text{(parallel ``or'')}$$

Proposition (Characterizing filters)

1. A separator $S \subseteq \mathscr{A}$ is a filter iff $\pitchfork^{\mathscr{A}} \in S$
2. A classical separator $S \subseteq \mathscr{A}$ is a filter iff $\text{p-or}^{\mathscr{A}} \in S$

Finitely generated separators and principal filters

Theorem

Given a separator $S \subseteq \mathscr{A}$, the following are equivalent:

1. S is finitely generated and $\pitchfork^{\mathscr{A}} \in S$
2. S is a principal filter: $S = \uparrow\{\Theta\}$ for some $\Theta \in S$

   (Θ is called the universal proof of S)
3. The induced Heyting algebra $\mathscr{A}/S$ is complete, and the canonical surjection $[\cdot] : \mathscr{A} \to \mathscr{A}/S$ commutes with infinitary meets:

$$\Big[\bigcurlywedge_{i\in I} a_i\Big] = \bigwedge_{i\in I}[a_i]$$

In model theoretic terms, this situation corresponds to a collapse of (intuitionistic/classical) realizability into (Kripke/Cohen) forcing!

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The implicative tripos (1/2)

Let $(\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

- For each set I, we observe that:
  - The triple $\mathscr{A}^I = (\mathscr{A}^I, \preccurlyeq^I, \to^I)$ is an implicative structure, whose ordering $\preccurlyeq^I$ and implication $\to^I$ are defined componentwise (power implicative structure)
  - The set of constant I-indexed families in S generates a separator

$$S[I] \;:=\; \big\{(a_i)_{i\in I} \in \mathscr{A}^I : (\exists s\in S)(\forall i\in I)\, s \preccurlyeq a_i\big\} \;\subseteq\; \mathscr{A}^I$$

(uniform power separator)

So that we can let $\mathbf{P}(I) \;:=\; \mathscr{A}^I/S[I]$ (induced Heyting algebra)

Theorem (Implicative tripos)

1. The correspondence $I \mapsto \mathbf{P}(I)$ is functorial (in a contravariant way)
2. The functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a tripos

Recall: Tripos = categorical model of higher-order logic

The implicative tripos (2/2)

- The above construction encompasses many well-known triposes:
  - Forcing triposes, which correspond to the case where $(\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra, and $S = \{\top\}$ (i.e. no quotient)
  - Triposes induced by total combinatory algebras... (int. realizability)
    ... and even by partial combinatory algebras, via some completion trick
  - Triposes induced by abstract Krivine structures (class. realizability)
- As for any tripos, each implicative tripos can be turned into a topos via the standard tripos-to-topos construction
- **Question:** What do implicative triposes bring new w.r.t.
  - Forcing triposes (intuitionistic or classical)?
  - Intuitionistic realizability triposes?
  - Classical realizability triposes?

Characterizing some implicative triposes

Theorem (Characterizing forcing triposes)

Let $\mathbf{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{HA}$ be the tripos induced by an implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$. Then the following are equivalent:

1. The tripos $\mathbf{P}$ is isomorphic to a forcing tripos
2. The separator $S \subseteq \mathscr{A}$ is a principal filter of $\mathscr{A}$
3. The separator $S \subseteq \mathscr{S}$ is finitely generated and $\pitchfork^{\mathscr{A}} \in S$

Slogan: Forcing $=$ non-deterministic realizability

Theorem (Classical implicative triposes)

Each tripos induced by a classical implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$ is isomorphic to a tripos induced by an abstract Krivine structure

Classical implicative algebras $\sim$ Abstract Krivine Structures (same expressiveness)

Higher-order completeness (1/2)

Implicative triposes encompass all the well-known (intuitionistic/classical) forcing & realizability triposes

But do they encompass all triposes?

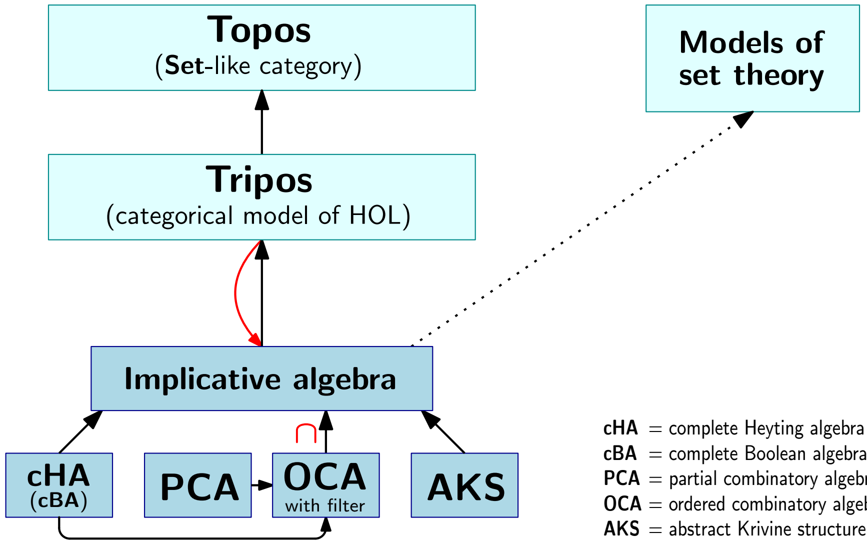
Theorem (Higher-order completeness/Representation)

Each $\mathbf{Set}$-based tripos is (isomorphic to) an implicative tripos

Note: From the point of view of foundations, the above theorem expresses that a whole tripos (= structured proper class) can be described by a single implicative algebra (= structured set) $\Rightarrow$ Reduction of complexity

- Explains *a fortiori* why we succeeded to turn well-known triposes (induced by HAs, OCAs, AKSs, etc.) into implicative triposes
- Since implicative algebras have the same expressiveness as OCAs with filters, the completeness theorem also holds for the latter

Higher-order completeness (2/2)



cHA = complete Heyting algebra
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First-order completeness (1/2)

Implicative algebras can be used to interpret 1st-order theories as well

- Given an implicative algebra $\mathscr{A}$, define the notion of $\mathscr{A}$-model of a 1st-order language $\mathscr{L}$ (resp. of a 1st-order theory $\mathscr{T}$) as expected
- Implicative model $=$ $\mathscr{A}$-model for some implicative algebra $\mathscr{A}$

Proposition (Soundness)

If $\mathscr{T} \vdash \phi$, then $\mathscr{M} \models \phi$ in all implicative models $\mathscr{M}$ of $\mathscr{T}$

Theorem (Strong completeness for implicative models) [M. 2022]

For each classical 1st-order theory $\mathscr{T}$, there is a full implicative model $\mathscr{M}$ (over some classical implicative algebra) that captures $\mathscr{T}$:

$$\mathscr{T} \vdash \phi \qquad \text{iff} \qquad \mathscr{M} \models \phi \qquad (\phi \text{ closed})$$

- Strong completeness theorem already holds for Boolean-valued models, but the proof relies on the completeness theorem of 1st-order logic

First-order completeness (2/2)

Let $\mathscr{T}$ be a consistent (classical) 1st-order theory

- From the strong completeness theorem, there is a full implicative model $\mathscr{M}$ (over some classical implicative algebra $\mathscr{A}$) such that:

  $$\mathscr{T} \vdash \phi \qquad \text{iff} \qquad \mathscr{M} \models \phi \qquad\qquad (\phi \text{ closed})$$

  Moreover the implicative algebra $\mathscr{A}$ is consistent since the theory $\mathscr{T}$ is

- Picking some ultraseparator $U \supseteq S_{\mathscr{A}}$, get a Tarski model $\mathscr{M} : U$:

  $$\mathscr{T} \vdash \phi \qquad \text{implies} \qquad \mathscr{M}{:}U \models \phi \qquad\qquad (\phi \text{ closed})$$

Therefore we get:

Factorization of 1st-order completeness

FO-theory		**Impl. model**		**Tarski model**
$\mathscr{T} \vdash \phi$	$\Longleftrightarrow$ (constructive)	$\mathscr{M} \models \phi$	$\overset{U}{\Longrightarrow}$ (non constr.)	$\mathscr{M}{:}U \models \phi$

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Conclusion

Implicative algebra $=$ an algebraic structure to factorize model-theoretic constructions underlying forcing and realizability (intuitionistic & classical)

- **Idea:** Truth values can be manipulated as generalized realizers

Proof = Program = Type = Formula

- Each implicative algebra induces an implicative tripos, and this correspondence is surjective (up to isomorphism)
- In this structure: forcing = non deterministic realizability
- Classical implicative algebras $\sim$ Abstract Krivine Structures

Ongoing work:

- Conjunctive & disjunctive algebras [Miquey '20]
- Evidenced Frames [Cohen-Miquey-Tate '22]
- The category of implicative algebras: which notion of morphism?
- Implicative models of (I)ZF set theory