Philosophical perspectives on category theory

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Outline of talk

Some philosophical questions

- *What is there?* Metaphysics
- *How do we know?* Epistemology
- *How should we think?* Logic.
- *What should we do?* Ethics/Political Philosophy
- ...

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Plenty of scope to set category theory to the task of answering some of these questions.

Since category theory arose within mathematics, it's natural to wonder what this says about mathematics.

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I was always far more interested in the latter path, so thought I'd found heaven when back in 1995 I encountered a mathematical physicist writing ...

John Baez and This Week's Finds

The other, called the "The Tale of n-Categories", was an introduction to categories and higher categories — mainly just 2-categories. It can be found in these issues:

- Week 73 - The category of sets and the 2-category of categories.
- Week 74 - Kinds of categories: monoids, groups, and groupoids. The periodic table of n-categories.
- Week 75 - The fundamental groupoid of a topological space. The classifying space of a groupoid.
- Week 76 - Equations, isomorphisms, and equivalences. Adjoint functors.
- Week 77 - Adjoint functors.
- Week 78 - Adjoint functors and adjoint linear operators.
- Week 79 - The unit and counit of an adjunction.
- Week 80 - The definition of 2-category.
- Week 83 - Adjunctions in 2-categories and dual objects in monoidal categories.
- Week 84 - Review. Monads and monoids.
- Week 89 - Monads in 2-categories. Monoids and monoidal categories as monads.
- Week 92 - Monads from adjunctions.
- Week 99 - 2-Hilbert spaces. Coproducts.
- Week 100 - Definitions of n-category.

My project, c. 2000

- What do we learn from the ongoing expansion of (higher) category theory in mathematics and physics?
- What does this shift of foundational language amount to? Have we just been dealing with shadows which we now need to categorify? Why the string diagrams? What is higher gauge theory? Etc.

> Our proof of Theorem B is entirely elementary. We analyse the numerical properties of the sequence of numbers $\zeta_X(n) := |[\mathbb{T}^n, X]|$ related to the Euler-Morava characteristics of X only via Lurie's formula, which we treat as a black-box. Nevertheless, we take the opportunity to provide some theoretical context, that might point towards deeper structural phenomena of which the results of this paper are mere numerical shadows (and perhaps also to stronger versions thereof). As this is solely for expository purposes, we merely sketch the relevant ideas in an informal way. More information can be found in [HL13, CSY22, CSY21, Lur19].

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Lior Yanovski, Homotopy Cardinality via Extrapolation of Morava-Euler Characteristics

- As an experiment, in 2006 I posed the question of how to categorify Klein geometry.
- Instead of quotients of groups by stabilizers (Erlangen program), consider 2-groups and their actions.

Categorification of geometry

My proposed categorification of Klein geometry:

> Either way what prevents a Erlangen program for 2-groups?

Nothing! Especially since the Erlangen program is just the flip side of Galois theory: (see especially the slide about the icosahedron), and Galois theory has already been n-categorified to powerful and still growing effect.

But you're right - nobody seems to have thought hard about Klein geometry with Lie 2-groups (or higher) replacing Lie groups. Somehow people have skipped straight to categorifying principal bundles, even though principal bundles are a stripped-down way of thinking about Cartan geometries, which generalize Klein geometries! Sometimes ontogeny fails to recapitulate phylogeny. So, maybe the "punters" should be handed a nice specific Lie 2-group, some 2-spaces on which it acts, and be asked to study the "incidence relations" between these figures. Incidence geometry could be given a whole new lease on life!

Best, jb

Categorification of geometry

September 4, 2006

Klein 2-Geometry V

Posted by David Corfield

I had hoped to mark my first appearance in the Café with a striking new contribution to our Klein 2-geometry project. The **project** began on my old blog back in May, and you can follow it through its twists and turns over the next 3 monthly instalments. I have enjoyed both participating in a mathematical dialogue and, as a philosopher, thinking about what such participation has to do with a theory of enquiry. The obvious comparison for me is with the fictional dialogue *Proofs and Refutations* written by the philosopher Imre Lakatos in the early 1960s. The clearest difference between these two dialogues is that Lakatos takes the engine of conceptual development to be a process of

conjectured result (perhaps imprecisely worded) - proposed (sketched) proof - suggested counterexample - analysis of proof for hidden assumptions - revised definitions, conjecture, and improved proof,

whereas John, I and other contributors look largely to other considerations to get the concepts ‘right’. For instance, it is clear that one cannot get very far without a heavy dose of analogical reasoning, something Lakatos ought to have learned more about from Polya, both in person and through his books.

Categorification of geometry

Klein 2-geometry and its generalisation to higher Cartan geometry later bore fruit, in particular by way of Urs Schreiber's work in physics.

> It is therefore maybe curious to note that while Cartan geometry as originating in (Cartan 23) drew its motivation from the mathematical formulation of the theory of Einstein gravity, higher Cartan geometry is well motivated by higher dimensional supergravity such as 10d type II supergravity and heterotic supergravity as well as 11-dimensional supergravity.

Categorification of logic

March 6, 2008

Worrying About 2-Logic

Posted by David Corfield

Here's a possible problem for the idea of modal logic as 2-logic.

In ordinary first order logic a model of a theory is a set X. To an n-ary predicate of the theory we assign a subset of X^n, to a constant an element of X, and so on.

For a given X, we can derive a Galois correspondence between theories modelled on X and subgroups of $X!$, the permutations of X, as Todd **shows**.

Now, in first order modal logic (FoS4) a model of a theory is a sheaf. To show completeness we can stick with bog standard sheaves on topological spaces, as Awodey and Kishida show in their paper **Topology and Modality**. This combines the topological semantics of propositional modal logic with the set-valued semantics of first-order logic. Necessity relates to taking the interior of subsets of the base space.

February 25, 2010

(Infinity, 1)-logic

Posted by David Corfield

We're having a chat over **here** about what an $(\infty, 1)$-logic might look like. The issue is that if we can extract a (1)-logic from ordinary toposes, shouldn't there be an **$(\infty, 1)$-logic** to be extracted from $(\infty, 1)$-toposes. This post originated as an ordinary comment, but as things have gone a little quiet at the Café (10 days without a post!), I thought I'd promote it.

Won't there be a sense in which this internal logic to an $(\infty, 1)$-topos will have to be interpretable as a 'logic of space'? If Set is an especially nice topos which being **well-pointed** allows us to understand 1-logic internally and externally, we might hope that the well-pointed $(\infty, 1)$-topos of ∞-groupoids, ∞Gpd, does the same for $(\infty, 1)$-logic, given ∞-groupoids are models for spaces.

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- By 2012, *Homotopy Type Theory* emerged, a structural logic.
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- By 2012, *Homotopy Type Theory* emerged, a structural logic.
- Forget the scary looking '$(\infty, 1)-$' appearing everywhere – things are made simpler by not imposing uniqueness of identity proofs.
- The boundary of logic and mathematics is altered, e.g., $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/2\mathbb{Z}$ is part of the 'logic'.

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> To conclude, let me say a bit about how this project relates to modal logic. In a syntactic 2-theory with multiple generating types, the objects of the resulting semantic 2-category are not single structured categories, but diagrams of several categories with functors and natural transformations between them. Thus, the corresponding syntactic 1-theories have several "classes" of types, one for each category. These classes of types are generally called "modes", type theory or logic with multiple modes is called "modal", and the functors between these categories are called "modalities". Thus, modal logics are particular 2-theories, to which our framework applies.

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- There's still so much going on here, cf. Mike's recent Semantics of multimodal adjoint type theory.

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- New logic for a new mathematics for a new physics.
- Modal HoTT to allow the expression of twisted equivariant differential cohomology on superorbifolds to allow the formulation of M-theory, 'Hypothesis H'.
- For a discussion of why we need to produce the rational trajectory to this moment, see my Thomas Kuhn, Modern Mathematics and the Dynamics of Reason.

Mathematical prediction

There are very many consequences arising from this program: anomaly cancellation in M-theory, mysterious triality, Topological Data Analysis, etc., and even expectations of where the Cayley distance kernel is positive semidefinite.

arXiv > math > arXiv:2105.02871

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Mathematics > Geometric Topology

[Submitted on 6 May 2021 (v1), last revised 16 Sep 2021 (this version, v2)]

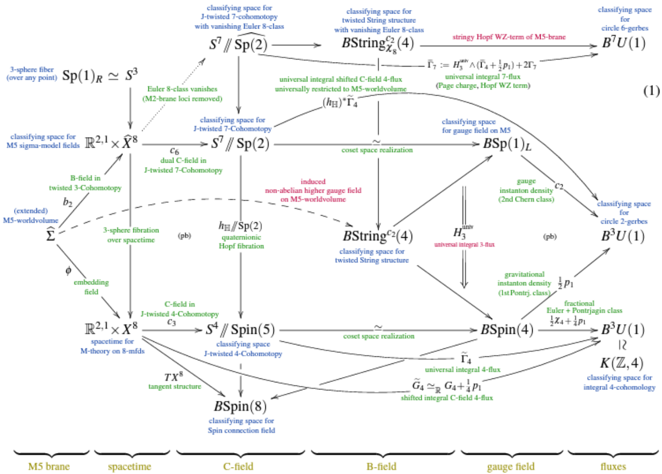
Fundamental weight systems are quantum states

David Corfield, Hisham Sati, Urs Schreiber

Weight systems on chord diagrams play a central role in knot theory and Chern-Simons theory; and more recently in stringy quantum gravity. We highlight that the noncommutative algebra of horizontal chord diagrams is canonically a star-algebra, and ask which weight systems are positive with respect to this structure; hence we ask: Which weight systems are quantum states, if horizontal chord diagrams are quantum observables? We observe that the fundamental gl(n)-weight systems on horizontal chord diagrams with N strands may be identified with the Cayley distance kernel at inverse temperature beta=ln(n) on the symmetric group on N elements. In contrast to related kernels like the Mallows kernel, the positivity of the Cayley distance kernel had remained open. We characterize its phases of indefinite, semi-definite and definite positivity, in dependence of the inverse temperature beta; and we prove that the Cayley distance kernel is positive (semi-)definite at beta=ln(n) for all n=1,2,3,... In particular, this proves that all fundamental gl(n)-weight systems are quantum states, and hence so are all their convex combinations. We close with briefly recalling how, under our "Hypothesis H", this result impacts on the identification of bound states of multiple M5-branes.

Comments: 21 pages; 1 figure; v2: bound sharpened
Subjects: **Geometric Topology (math.GT)**; High Energy Physics - Theory (hep-th); Mathematical Physics (math-ph); Combinatorics (math.CO); Quantum Physics (quant-ph)
Cite as: arXiv:2105.02871 **[math.GT]**
(or arXiv:2105.02871v2 **[math.GT]** for this version)
https://doi.org/10.48550/arXiv.2105.02871

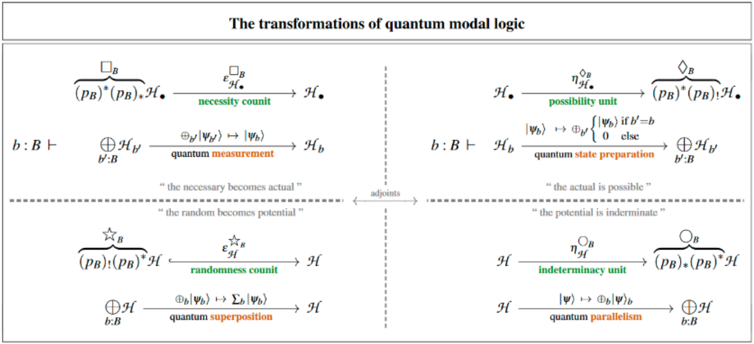
General and particular aspects of Hypothesis H



Why 4, 7, 11, etc? Sometimes we can speak category-theoretically about the particular (e.g., the free such-and-such), but let's turn to the general.

Quantum modal logic

One recent body of work coming out of this work involves a turn to topological quantum computing and to a quantum modal logic



Philosophy of physics potential

At the Center for Quantum and Topological Systems they're devising a quantum programming language, QS:

> Besides these technical properties, the logical language QS is curiously satisfying on quantum-philosophical grounds: For example, the internal language-construct in QS for quantum measurement via the modal logic of necessity is *verbatim* the same as for classical measurement, only now applied to (dependent) linear types where it happens to *imply* the collapse of the wavefunction in the categorical semantics. But in the internal logic this effect is just standard conditioning of expectations. In this sense the notorious "measurement problem" of quantum physics disappears when we speak proper QS. (This is analogous to what happens internal to proper quantum probability theory, see there.) Moreover, the *deferred measurement principle* verified in QS implies that even this collapse of the wavefunction in subsystems may be arbitrarily postponed, by observers who have access to the system at large (the "bath").

Exciting possibility to think about how quantum physics differs from classical physics, how one might conceptualise its emergence.

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- Applied Category Theory today studies: *causality, probabilistic reasoning, statistics, learning theory, dynamical systems, information theory, database theory, natural language processing, cognition, consciousness, systems biology, complex networks, game theory, robotics, quantum computing, ...*
- Understandable wish to promote new fields of application of category theory, but I doubt that there's any natural separation from the computer science/ physics applications.

Category theory allowing knowledge transfer

...we should treat the use of categorical concepts as a natural part of transferring and integrating knowledge across disciplines. The restructuring employed in applied category theory cuts through jargon, helping to elucidate common themes across disciplines. (ACT 2018)

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Technical tools that are typically employed in analytic philosophy are first-order logic and modal logic.

Now we have an opportunity to redo what's done but with a different and much richer toolkit, one that continues to evolve in a highly-principled manner. ACT is already contributing.

Approaches to mathematics, recalled with HoTT

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- Good to keep a 'dynamic' attitude in mind, however. Who knows what comes next, a directed HoTT? (Note that a lecture series on *Univalent Directed Type Theory* at CMU begins next week.)

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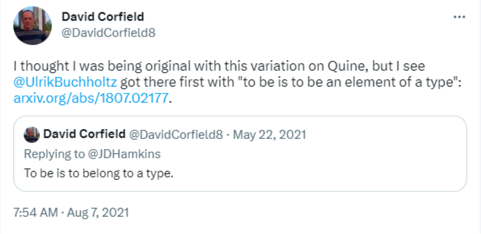

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- Gilbert Ryle's category mistakes

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- Robert Brandom's inferentialism, types as making explicit preservation of meaningfulness.

Natural language and type theory/category theory

- Aarne Ranta on *Type-theoretic grammar*
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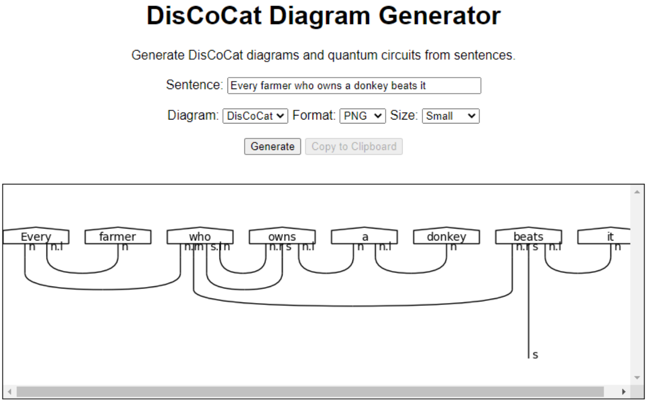
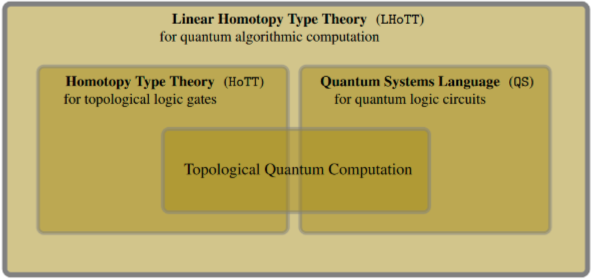


Natural language and type theory/category theory

- Linear HoTT as a synthesis?



Lawvere on quantifiers

For **H** a topos (or ∞-topos) and $f : X \to Y$ an arrow in **H** induces a 'base change', f^*, between slices (categories of dependent types):

$$(\sum_f \dashv f^* \dashv \prod_f) : \mathbf{H}/X \underset{f_*}{\overset{\overset{f_!}{\to}}{\underset{\overset{f^*}{\leftarrow}}{\to}}} \mathbf{H}/Y$$

This base change has dependent sum and product as left and right adjoint.

Common technique

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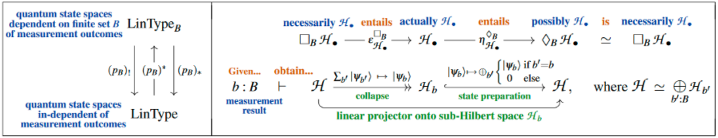
- Behavioral mereology: $A \leftarrow C \to B$ to express constraints, allowance and ensurance.
- Equivariance can be built in.
- Integral transforms: push-pull.

Possible worlds and many worlds

From all this we find, first of all, that

1. the necessity-counit exhibits the *collapse* of quantum states under a quantum measurement in the B-basis
2. the possibility-unit exhibits the condition quantum state preparation in this basis:



Remark 3.1. The collapse-principle of quantum measurement accurately expressed by the above linear necessity → actuality → possibility map was maybe first highlighted and made explicit in von Neumann (1932), §III.3, at least for the special case that (in our notation here) $\mathcal{H}_\bullet = (p_B)^* \mathbb{C}$ (cf. Lüders (1951)).

Remark 3.2. We see that the base type B (4) *of quantum measurement outcomes* plays the role of *both*

1. the *possible worlds* of classical modal logicians;
2. the *many worlds* of quantum interpretationists.

(...)

nLab

Conclusions

- Category theory continues to reveal its enormous heuristic power.
- Its use in mathematics and physics should change our philosophical understanding of these fields in terms of what they are about and where they're aiming.
- Constructions in new applied areas are of just the kind to interest philosophy (systems, causality, learning,...).
- To any young philosopher listening who wonders whether to learn some category theory: I'm glad I did. Things should be easier for you now, but never underestimate the inertia.