Dual algebraic structures and enrichment

Christina Vasilakopoulou

National Technical University of Athens, Greece

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Outline

1. Background
2. Sweedler theory for (co)monoids and (co)modules
3. Many-object generalization
4. Further directions

Algebras and coalgebras

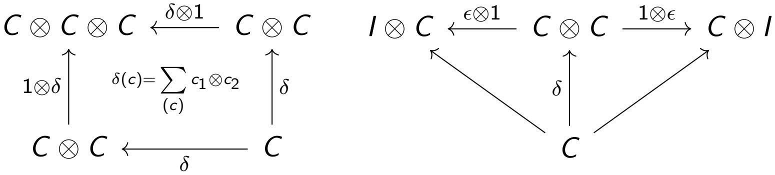
Suppose $(\mathcal{V}, \otimes, I)$ is monoidal category.

- A *monoid* is an object A together with maps $\mu\colon A \otimes A \to A$ and $\eta\colon I \to A$ which are associative and unital:

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu\otimes 1} & A \otimes A \\
{\scriptstyle 1\otimes\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
A \otimes A & \xrightarrow{\quad\mu\quad} & A
\end{array}
\qquad
\begin{array}{ccccc}
I \otimes A & \xrightarrow{\eta\otimes 1} & A \otimes A & \xleftarrow{1\otimes\eta} & A \otimes I \\
& \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\
& & A & &
\end{array}
$$

$\star$ In $(\mathrm{Ab}, \otimes, \mathbb{Z})$, rings; in $(\mathrm{Vect_k}, \otimes, k)$, k-algebras; in $(\mathrm{Cat}, \times, \mathbf{1})$, *strict* monoidal categories!

- A *comonoid* is an object C together with maps $\delta\colon C \to C \otimes C$ and $\epsilon\colon C \to I$ which are coassociative and counital:



⋆ In $(\mathrm{Cat}, \times, \mathbf{1})$, any category! With $\delta(X)=(X,X)$ and $\epsilon(X)=*$ "trivially".

⋆ In $(\mathrm{Mod}_R, \otimes_R, R)$, R-coalgebras: divided power coalgebra, tensor algebra, group-like coalgebra, trigonometric coalgebra...

> Monoids and comonoids in $(\mathcal{V}, \otimes, I)$, together with maps that preserve (co)multiplication and (co)units, form categories Mon and Comon.

Suppose $(\mathcal{V}, \otimes, I)$ is symmetric, with $\sigma_{XY} \colon X \otimes Y \cong Y \otimes X$.

Mon and Comon are themselves monoidal, with I and

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \sigma \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\mu \otimes \mu} A \otimes B$$

Suppose $(\mathcal{V}, \otimes, I, \sigma)$ is monoidal closed, with $- \otimes X \dashv [X, -]$ for all X.

For any comonoid C and monoid A, $[C, A]$ is a monoid via *convolution*

$$[C, A] \otimes [C, A] \xrightarrow{*} [C, A] \text{ which under tensor-hom adj. is}$$

$$[C,A] \otimes [C,A] \otimes C \xrightarrow{1 \otimes \delta} [C,A] \otimes [C,A] \otimes C \otimes C \xrightarrow{1 \otimes \sigma \otimes 1} [C,A] \otimes C \otimes [C,A] \otimes C \xrightarrow{\mathrm{ev} \otimes \mathrm{ev}} A \otimes A \xrightarrow{\mu} A$$

$$(f * g)(c) = \sum_{(c)} f(c_1) g(c_2)$$

Sweedler theory: Motivation

Idea 0: For any k-coalgebra C, its linear dual $C^* = \mathrm{Hom}_k(C, k)$ is a k-algebra via convolution. For any k-algebra A, A^* is a coalgebra only when it is finite-dimensional. Find an operation that 'fixes' that?

Idea 1: [Sweedler, 1969] For any three vector spaces A, B and C,

$$\mathrm{Hom}(C \otimes B, A) \cong \mathrm{Hom}(B, \mathrm{Hom}(C, A)).$$

If C coalgebra, A, B algebras, when is it an *algebra* map $B \to \mathrm{Hom}(C, A)$?
Answer (low-level): when $f\colon C \otimes B \to A$ *measures*, i.e. satisfies

$$f(c \otimes aa') = \sum f(c_{(1)} \otimes a) f(c_{(2)} \otimes a')$$
$$f(c \otimes 1) = \epsilon(c)1$$

There exists a *universal measuring* coalgebra P, namely for any other measuring coalgebra C, we get a unique coalgebra map $C \to P$.

$\star$ Constructed bijection from $\mathsf{Alg}(B, \mathrm{Hom}(C, A))$ to $\mathsf{Coalg}(C, P)$, where $P{=}P(A, B)$ is sum of subcoalgebras of cofree coalgebra on $\mathrm{Hom}(A, B)$...

Answer (high-level): $\mathrm{Hom}(\text{-}, A) \colon \mathsf{Coalg}^{\mathrm{op}} \to \mathsf{Alg}$ has an adjoint $P(A, \text{-})$!

Special case of more general result. . . for *locally presentable* categories.

A category is locally presentable when it has all colimits, and all objects are (λ-)filtered colimits of a set of certain presentable objects.

$\star$ From Vect_k, move to (d)gVect, Mod_R and many more!

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. There is a 'parameterized' adjunction between

$$[\text{-}, \text{-}] \colon \mathsf{Comon}^{\mathrm{op}} \times \mathsf{Mon} \to \mathsf{Mon} \quad \text{given - convolution}$$

$$P(\text{-}, \text{-}) \colon \mathsf{Mon}^{\mathrm{op}} \times \mathsf{Mon} \to \mathsf{Comon} \quad \text{new - universal measuring}$$

Sweedler theory for monoidal categories

- [Anel-Joyal, 2013] : dgVect_k, functors related to bar-cobar construction
  - convolution $[\text{-},\text{-}]$ and 'Sweedler hom' $P(\text{-},\text{-})$
  - 'Sweedler product' $N(\text{-},\text{-})\colon \mathrm{Comon}\times\mathrm{Mon}\to\mathrm{Mon}$ w. $N(C,\text{-})\dashv[C,\text{-}]$
- In fully general setting, universal measuring comonoid is

$$P(A,B)=\left(\mathrm{Lan}_{[\text{-},B]}\mathbf{1}_{\mathrm{Comon}}\right)(A)=\int^{C}\mathrm{Mon}(A,[C,B])\cdot C$$

⋆ For $\mathcal{V}$=Set, Comon ≅ Set and the set $P(A,B)$ is $\mathrm{Mon}(A,B)$.

⋆ Low-level is special case $\mathcal{V}=\mathrm{Vect}_k$, also Idea 0: *finite Sweedler dual*

$$P(A,k)=A^{o}=\{f\in A^{*}\mid \mathit{kerf}\ \text{contains cofinite ideal}\}$$

for which $\mathrm{Alg}(A,C^{*})\cong\mathrm{Coalg}(C,A^{o})$.

⋆ 'Generalized algebra maps': $P(A,B)$ contains k-algebra maps, as the group-like elements $\delta(f)=f\otimes f$.

Enrichment of algebras in coalgebras

[Wraith, 1970's] k-algebras are enriched in k-coalgebras...

- Induced convolution $[\text{-},\text{-}]$ has extra structure: it is an *action* of the monoidal category $\mathrm{Comon}^{(\mathrm{op})}$ on the category Mon!

A $(\mathcal{V},\otimes,I)$-action on $\mathcal{C}$ is some $\bullet\colon \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ with $(V\otimes W)\bullet C\cong V\bullet(C\bullet V)$ and $I\bullet C\cong C$ plus usual axioms.

Any parameterized adjoint of an action $\bullet\colon \mathcal{V}\times\mathcal{C}\to\mathcal{C}$ gives rise to a $\mathcal{V}$-enriched structure on $\mathcal{C}$; all 'tensored' $\mathcal{V}$-categories arise this way.

Desired enrichment in very general setting, putting all pieces together.

Suppose $\mathcal{V}$ is symmetric monoidal closed and locally presentable. The category Mon is enriched in the symmetric monoidal Comon.

Digression: theory of Hopf categories

- A *bimonoid* in $\mathcal{V}$ is a monoid and a comonoid in a compatible way ; a *Hopf monoid* is a bimonoid with antipode.
- Their many-object generalizations? *Semi-Hopf* and *Hopf* $\mathcal{V}$-categories.

In particular, a semi-Hopf $\mathcal{V}$-category comes with

- $H(x,y)\otimes H(y,z)\xrightarrow{m_{xyz}}H(z,x),\ I\xrightarrow{j_x}H(x,x)$ 'global' multipl
- $H(a,b)\xrightarrow{d_{ab}}H(a,b)\otimes H(a,b),\ H(a,b)\xrightarrow{e_{ab}}I$ 'local' comultipl

$$\begin{array}{ccc}
H_{x,y}\otimes H_{y,z} & \xrightarrow{d_{xy}\otimes d_{yz}} & H_{x,y}\otimes H_{x,y}\otimes H_{y,z}\otimes H_{y,z} \\
 & & \downarrow{\scriptstyle 1\otimes\sigma\otimes 1} \\
{\scriptstyle m_{xyz}}\downarrow & & H_{x,y}\otimes H_{y,z}\otimes H_{x,y}\otimes H_{y,z} \\
 & & \downarrow{\scriptstyle m_{xyz}\otimes m_{xyz}} \\
H_{x,z} & \xrightarrow[d_{xz}]{} & H_{x,z}\otimes H_{x,z}
\end{array}$$

The category of monoids is a semi-Hopf $\mathcal{V}$-category.

Universal measuring comodules

[Batchelor, 1990's] (Universal) measuring comodules: motivation and construction follows Sweedler's, applications to algebra and geometry

⋆ Sketch high-level approach, involving *(co)modules* in arbitrary $(\mathcal{V}, \otimes, I)$

- For a monoid A, an A-module M comes with $\mu\colon A \otimes M \to M$ (associative and unital)
- For a comonoid C, a C-comodule X comes with $\chi\colon X \to C \otimes X$ (coassociative and counital)
- With maps preserving (co)actions, categories ${}_A\mathrm{Mod}$ and ${}_C\mathrm{Comod}$

■ 'Global' categories Mod, Comod of (co)modules for any (co)monoid, maps for Mod are $g\colon {}_AM \to {}_BN$ in $\mathcal{V}$ with $f\colon A \to B$ in $\mathrm{Mon}(\mathcal{V})$ that

$$\begin{array}{ccccc}
A \otimes M & & \xrightarrow{\quad\mu\quad} & & M \\
{\scriptstyle 1\otimes g}\downarrow & & & & \downarrow{\scriptstyle g} \\
A \otimes N & \xrightarrow{f\otimes 1} & B \otimes N & \xrightarrow{\mu} & N
\end{array}$$

$\star$ In a symmetric monoidal closed $\mathcal{V}$, for any C-comodule X, A-module M, $[X, M]$ is a $[C, A]$-module.

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. There is a parameterized adjunction between

$$[\text{-},\text{-}] \colon \mathsf{Comod}^{\mathrm{op}} \times \mathsf{Mod} \to \mathsf{Mod} \quad \text{given - convolution}$$

$$Q(\text{-},\text{-}) \colon \mathsf{Mod}^{\mathrm{op}} \times \mathsf{Mod} \to \mathsf{Comod} \quad \text{new - universal measuring}$$

$\star$ The functor $[\text{-},\text{-}]$ is again an action from Comod on Mod.

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. The category Mod is enriched in the symmetric monoidal Comod.

Enriched fibration

So far: enrichment of Mon in Comon, also of Mod in Comod

⋆ Independently of the enrichments, in <u>any</u> monoidal $\mathcal{V}$ these categories form a *fibration* Mod → Mon and an *opfibration* Comod → Comon

A fibration is a functor $F\colon \mathcal{C} \to \mathcal{D}$ with universal lifting property:

$$\begin{array}{cccccc} M' & \dashrightarrow & f^*M & \longrightarrow & M & \quad \text{in } \mathcal{C} \\ & & & & & \\ B' & \longrightarrow & A & \xrightarrow{\;f\;} & B & \quad \text{in } \mathcal{D} \end{array}$$

■ The general situation is captured by an *enriched fibration* structure

$$\begin{array}{ccc} \text{Mod} & \xrightarrow{\text{enriched}} & \text{Comod} \\ {\scriptstyle \text{fibration}}\big\downarrow & & \big\downarrow{\scriptstyle \text{opfibration}} \\ \text{Mon} & \xrightarrow{\text{enriched}} & \text{Comon} \end{array}$$

Generalizing from one to many objects

Initially: monoids become categories, comonoids become *cocategories*!

- If $(\mathcal{V}, \otimes, I)$ has coproducts preserved by $\otimes$, a $\mathcal{V}$-cocategory has objects $\mathrm{ob}\mathcal{C}$ and hom-objects $\mathcal{C}(x, y) \in \mathcal{V}$ with coherent

$$C(x,z) \xrightarrow{d_{xyz}} \sum_y C(x,y) \otimes C(y,z) \qquad C(x,x) \xrightarrow{\epsilon_x} I$$

Note: *opcategories*, i.e. $\mathcal{V}^{\mathrm{op}}$-categories, are not as convenient formally...

$\mathcal{V}$-categories are monads & $\mathcal{V}$-cocategories are comonads in *bicategory* of $\mathcal{V}$-matrices: objects are sets, maps $S\colon X \nrightarrow Y$ are families $\{S(x,y)\} \in \mathcal{V}$,

$$(S \circ T)(x,z) = \sum_y T(x,y) \otimes S(y,z) \quad \text{composition is matrix mult}$$

Under running assumptions, $\mathcal{V}$-Cocat has good properties (sym mon closed, loc pres) that allow "universal measuring cocategories"

Suppose $\mathcal{V}$ is a symmetric monoidal closed and locally presentable category. There is a parameterized adjunction between

$$H(\text{-},\text{-})\colon \mathcal{V}\text{-Cocat}^{\mathrm{op}} \times \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cat} \quad \text{given - `convolution'}$$
$$S(\text{-},\text{-})\colon \mathcal{V}\text{-Cat}^{\mathrm{op}} \times \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Cocat} \quad \text{new - universal measuring}$$

$H(C, A)$ consists of functions and $H(C,A)(f,g) = \prod_{x,y} [C(x,y), A(fx, gy)]$.

The category $\mathcal{V}$-Cat is enriched in the symmetric monoidal $\mathcal{V}$-Cocat.

Similar things happen for (co)modules for (co)categories...but behind technical results lies a clearer picture.

Generalizing from monoidal to double categories

Idea: clarify necessary structure on matrix double category & abstract!

- A double category $\mathbb{D}$ consists of
  - object category $\mathbb{D}_0$ (0-cells & vertical 1-cells)
  - arrow category $\mathbb{D}_1$ (horizontal 1-cells & 2-morphisms)
    $$\begin{array}{ccc} X & \overset{A}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & Y \\ f\downarrow & \Downarrow\alpha & \downarrow g \\ Z & \underset{B}{\longrightarrow\!\!\!\!\!\!\!+\;\;} & W \end{array}$$
  - $\mathbb{D}_0 \xrightarrow{\mathbf{1}} \mathbb{D}_1$, $\mathbb{D}_1 \underset{\mathfrak{t}}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathbb{D}_0$, $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\circ} \mathbb{D}_1$ plus coherent isomorphisms.

0-cells, horizontal 1-cells, *globular* 2-morphisms make bicategory $\mathcal{H}(\mathbb{D})$.

$\star$ For $\mathbb{D}=\mathcal{V}\text{-}\mathbb{M}\mathrm{at}$, $\mathcal{V}\text{-}\mathbb{M}\mathrm{at}_0=\mathrm{Set}$ & $\mathcal{H}(\mathcal{V}\text{-}\mathbb{M}\mathrm{at})$ the usual bicat of matrices

- A monad in $\mathbb{D}$ is $A\colon X \twoheadrightarrow X$ with associative, unital

$$\begin{array}{ccccc} X & \xrightarrow{A} & X & \xrightarrow{A} & X \\ \| & & \Downarrow\mu & & \| \\ X & & \xrightarrow[A]{\qquad\qquad} & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \| & \Downarrow\eta & \| \\ X & \xrightarrow[A]{} & X \end{array}$$

For any double category $\mathbb{D}$, there are categories of (co)monads $\mathsf{Mnd}(\mathbb{D})$, $\mathsf{Cmd}(\mathbb{D})$ as well as global categories of (co)modules $\mathsf{Mod}(\mathbb{D})$, $\mathsf{Comod}(\mathbb{D})$.

$\star$ $\mathsf{Mnd}(\mathcal{V}\text{-}\mathbb{M}\mathsf{at}) = \mathcal{V}\text{-}\mathsf{Cat}$ and $\mathsf{Cmd}(\mathcal{V}\text{-}\mathbb{M}\mathsf{at}) = \mathcal{V}\text{-}\mathsf{Cocat}$

- *Fibrant* double cats: vertical 1-cells turn to horizontal in a coherent way

$\star$ Function $f\colon X \to Y$ gives matrices $f^*(x,y) = f_!(y,x) = \begin{cases} I \text{ if } fx = y \\ 0 \text{ if } fx \neq y \end{cases}$

- *Monoidal* double cats: $\mathbb{D}_0$ and $\mathbb{D}_1$ monoidal, compatibly

$\star$ In matrices, $(X \otimes Y) = X \times Y$ & $(S \otimes T)((x,y),(z,w)) = S(x,z) \otimes T(y,w)$

- *Locally closed* monoidal double cats: $\mathbb{D}_0$ and $\mathbb{D}_1$ closed, compatibly

$\star$ For matrices, $[X, Y] = Y^X$ & $H(S, T)$ as before (not ad-hoc anymore!)

- *Locally presentable* double cats: $\mathbb{D}_0$ and $\mathbb{D}_1$ loc pres, compatibly

Sweedler theory for double categories

Suppose $\mathbb{D}$ is a locally closed symmetric monoidal double category, fibrant and locally presentable. Then there is a parameterized adjunction

$$H(\text{-},\text{-})\colon \mathsf{Cmd}(\mathbb{D})^{\mathrm{op}} \times \mathsf{Mnd}(\mathbb{D}) \to \mathsf{Mnd}(\mathbb{D}) \quad \text{given - `convolution'}$$

$$S(\text{-},\text{-})\colon \mathsf{Mnd}(\mathbb{D})^{\mathrm{op}} \times \mathsf{Mnd}(\mathbb{D}) \to \mathsf{Cmd}(\mathbb{D}) \quad \text{new - universal measuring}$$

Moreover, $\mathsf{Mnd}(\mathbb{D})$ is enriched in $\mathsf{Cmd}(\mathbb{D})$.

The fibration on the left is enriched in the opfibration on the right

$$\begin{array}{ccc} \mathsf{Mod}(\mathbb{D}) & \quad & \mathsf{Comod}(\mathbb{D}) \\ \downarrow & & \downarrow \\ \mathsf{Mnd}(\mathbb{D}) & & \mathsf{Cmd}(\mathbb{D}) \end{array}$$

⋆ A $\mathbb{D}$ with single object & vertical 1-cell 'is' a monoidal category...back to (co)monoids in monoidal categories!

Further directions

So far: established very broad framework for Sweedler theory – enrich monads in comonads, modules in comodules – in general double $\mathbb{D}$.

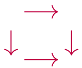
Next: employ or *extend* theory for interesting examples in other contexts!

- $\mathbb{D}$ =matrices gives universal measuring cocategories
- $\mathbb{D}$ =*symmetric sequences* give universal measuring cooperads...

$\mathbb{D}$	matrices	sequences
objects	sets $X, Y, \ldots$	sets $X, Y, \ldots$
vertical 1-cells	functions $f, g, \ldots$	functions $f, g, \ldots$
horizontal 1-cells $X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$	matrices $\{S_{x,y}\}$ i.e. $X \times Y \xrightarrow{S} \mathcal{V}$	symmetries i.e. $\Sigma(X) \times Y \xrightarrow{S} \mathcal{V}$
2-cells	$S_{x,y} \to T_{fx,gy}$	$S_{x_1,\ldots,x_n;y} \to T_{fx_1,\ldots,fx_n;gy}$

Goal: enrichment of (non-colored and colored) operads in cooperads, connect to bar-cobar construction and operadic Koszul duality!

Thank you for your attention!



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