How Categories Come to Matter

On the History and Sociology of Categories in Modern Mathematics

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TOPOS INSTITUTE COLLOQUIUM

THE UNIVERSITY OF EDINBURGH

Social History of Mathematics

"Some of you may have had occasion to run into mathematicians, and to wonder, therefore, how they got that way."

Tom Lehrer, "Lobachevsky," 1960

Social History of Mathematics

“how they got that way”

- What makes someone into a mathematician?
  - Culture, training, self-identity, perception, material and social support,…
- Who is included or excluded?
  - Who has power and authority?
  - How are power and authority distributed and used?
- What do mathematicians value?
  - How do they show and assess what they value?
  - How do values change what mathematicians do?
- How do mathematicians know what they know, and how do they share that knowledge?

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Main Claims Today

- **What do mathematicians value?**
  - Mathematicians value community and connection (both are moving targets).
  - Mathematicians' interest in (mathematical) categories and structures has developed to a significant extent from a 100+ year old interest in (personal and disciplinary) categories and structures.
- **How do mathematicians know what they know, and how do they share that knowledge?**

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- **How do mathematicians know what they know, and how do they share that knowledge?**
  - Sharing mathematics between people is hard. (E.g. blackboards.)
  - Sharing mathematics over long distances is harder. (E.g. articles and reviews.)
  - Material infrastructures affect not just *how* but *what* mathematicians share with each other.

Question

- What major foundational program in mathematics from the turn of the 20th century encountered a major disruptive development from an Austrian mathematician in the 1930s that would frame some of the defining foundational questions for the mid-20th century and beyond?


David Hilbert and Kurt Gödel,
Wikimedia Commons

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In universities, there are **no trick questions,** only ***surprisingly interesting questions.***

David Hilbert and Kurt Gödel, Wikimedia Commons

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  Hint: the major foundational program was the subject of a major address at one of the first International Congresses of Mathematicians.


David Hilbert and Kurt Gödel, Wikimedia Commons

Bibliography at the Fin-de-siècle

- International journals, travel, and community in late nineteenth century.
  - National organizations and infrastructures.
  - Most research supported and pursued in national contexts.

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Relatively recent idea: there is *an international community* of mathematicians.

Very recent idea: an international community *is the one that matters,* and should be prioritized.

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- Rudio, 1897 Zürich International Congress

  "… I would like to draw your attention to one point, perhaps **the most important** of all. The most important because it concerns a searing question whose solution requires energetic initiative: the **question of mathematical bibliography**."

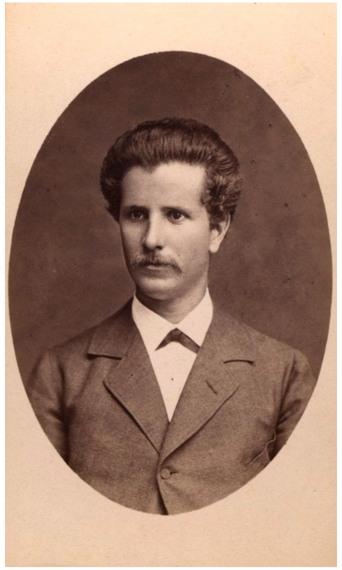
Bibliography at the Fin-de-siècle

- International journals, travel, and community in late nineteenth century.
  - National organizations and infrastructures.
  - Most research supported and pursued in national contexts.
- Rudio, 1897 Zürich International Congress
- Twinned problems: understanding the relationships among the **theories** and the **people and sites** of modern mathematics, ***and how to access them*****.**
- Common answer: publishing infrastructure and classification.

Question: continued

- What major foundational program in mathematics from the turn of the 20th century encountered a major disruptive development from an Austrian mathematician in the 1930s that would frame some of the defining foundational questions for the mid-20th century and beyond?
  - International bibliography.
  - 1930s disruption?

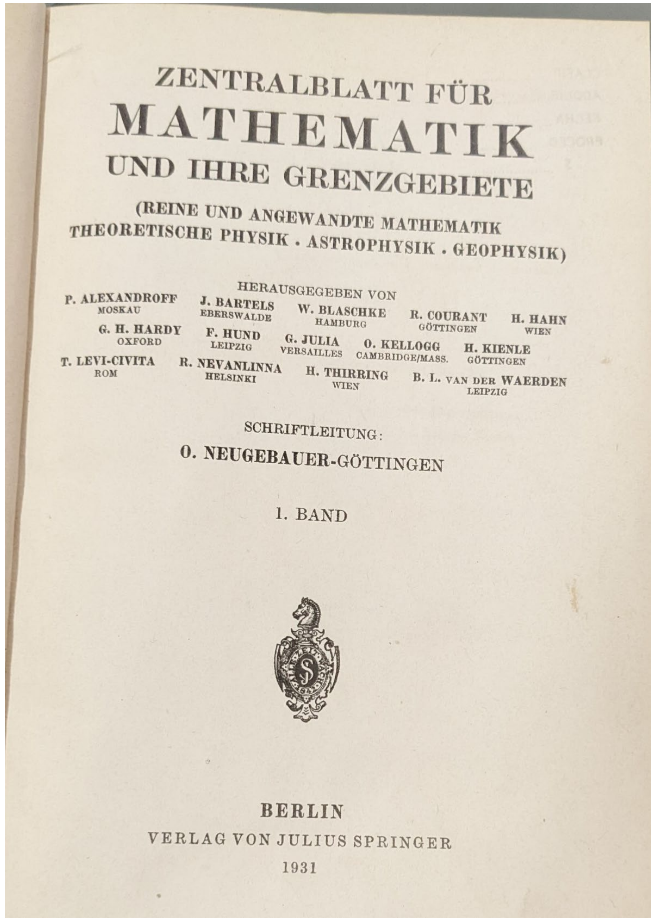
  Hint: the Austrian mathematician emigrated to the USA, joining Americans interested in this field to further transform the disruptive intervention, including relating to new technologies such as electronic computers.



Ferdinand Rudio and Kurt Gödel,
ETH-Bibliothek and Wikimedia Commons

Continuous Reviewing

- Changes reviews from national library resource to a tool of international commerce and community.
- Reviews as advertisements (for publishers and mathematicians).
- Uses networks and practices of well-connected outward-looking mathematicians.



ZENTRALBLATT FÜR
MATHEMATIK
UND IHRE GRENZGEBIETE
(REINE UND ANGEWANDTE MATHEMATIK
THEORETISCHE PHYSIK . ASTROPHYSIK . GEOPHYSIK)

HERAUSGEGEBEN VON
P. ALEXANDROFF MOSKAU · J. BARTELS EBERSWALDE · W. BLASCHKE HAMBURG · R. COURANT GÖTTINGEN · H. HAHN WIEN
G. H. HARDY OXFORD · F. HUND LEIPZIG · G. JULIA VERSAILLES · O. KELLOGG CAMBRIDGE/MASS. · H. KIENLE GÖTTINGEN
T. LEVI-CIVITA ROM · R. NEVANLINNA HELSINKI · H. THIRRING WIEN · B. L. VAN DER WAERDEN LEIPZIG

SCHRIFTLEITUNG:
O. NEUGEBAUER-GÖTTINGEN

1. BAND

BERLIN
VERLAG VON JULIUS SPRINGER
1931

Shelf after shelf of *Zentralblatt für Mathematik,* Instituto de Matématicas, UNAM

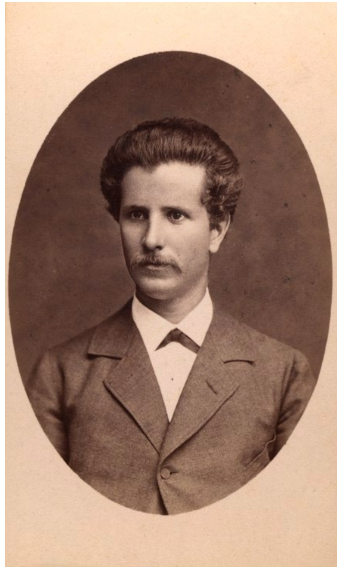
Question: answered

- What major foundational program in mathematics from the turn of the 20th century encountered a major disruptive development from an Austrian mathematician in the 1930s that would frame some of the defining foundational questions for the mid-20th century and beyond?
  - International Bibliography
  - Continuous International Reviewing

  …

  - Problems of mutual knowledge and connection

  across long distances.

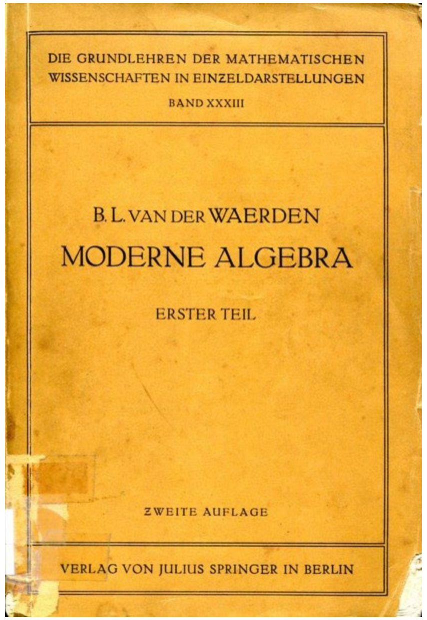


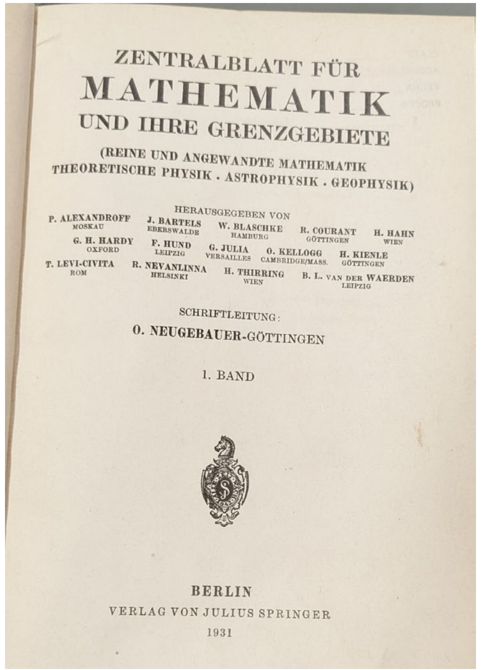

Ferdinand Rudio and Otto Neugebauer,
ETH-Bibliothek and MacTutor (St Andrews)

Springer’s Modernity

- Springer / van der Waerden, *Moderne Algebra,* 1930
  - Critical role of Emmy Noether!
- Springer / Neugebauer, *Zentralblatt für Mathematik,* 1931

- Two ways to organize modern mathematics.
- Based on mathematical relationships within an overall conceptual architecture.
- Prepares the way for viewing mathematics in terms of structural and categorical relations.

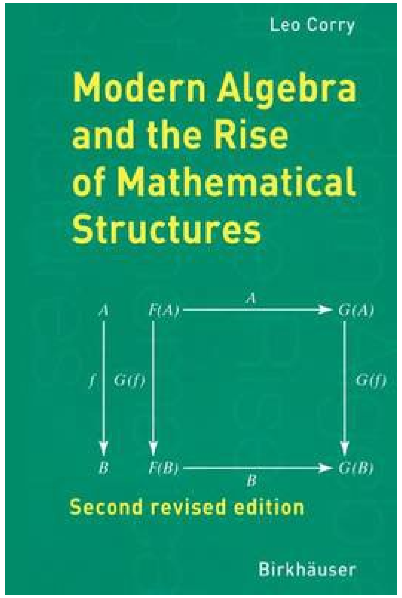


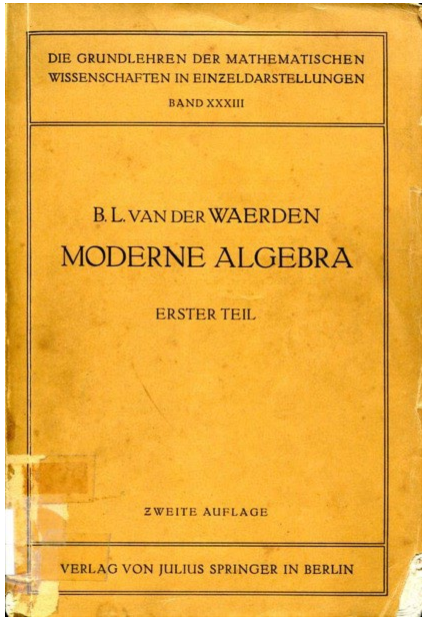


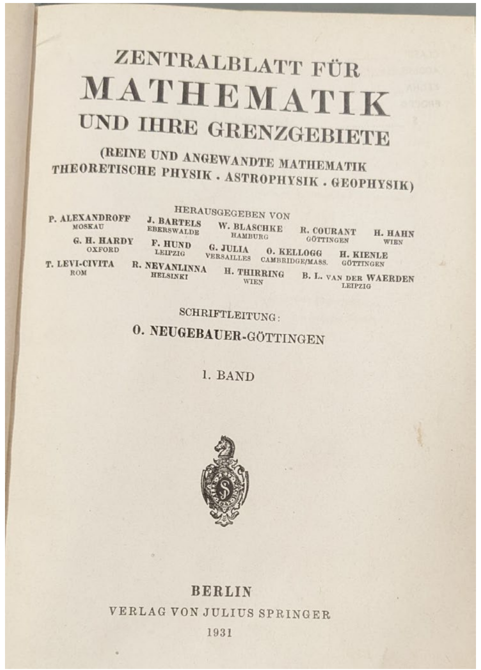
Springer's Modernity

- Argument of Leo Corry (1997):

Image of Mathematics	**Mathematical Structural*ism***
Body of Mathematics	**Mathematical Structures**







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  - Modified to include *Zentralblatt:*

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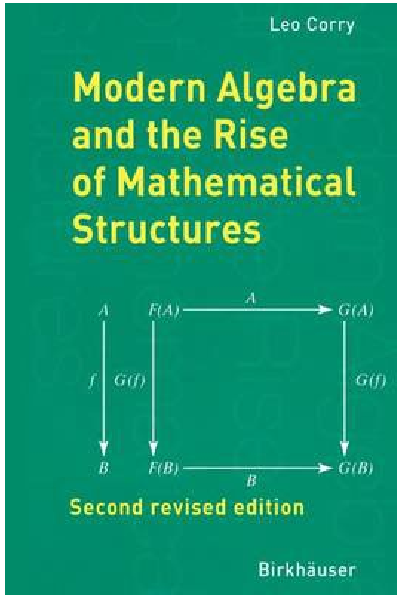

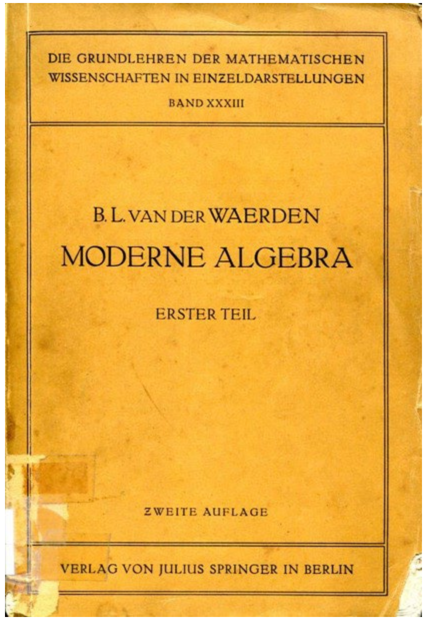

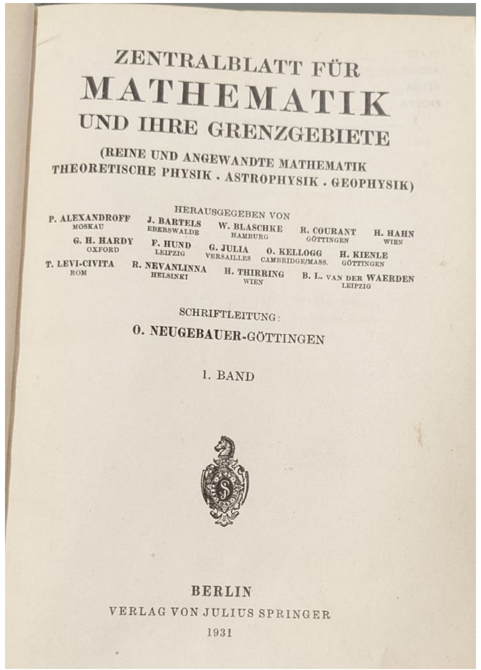

Springer's Modernity

- Argument of Leo Corry (1997)
  - Modified to include *Zentralblatt:*
    - What do these have in common?

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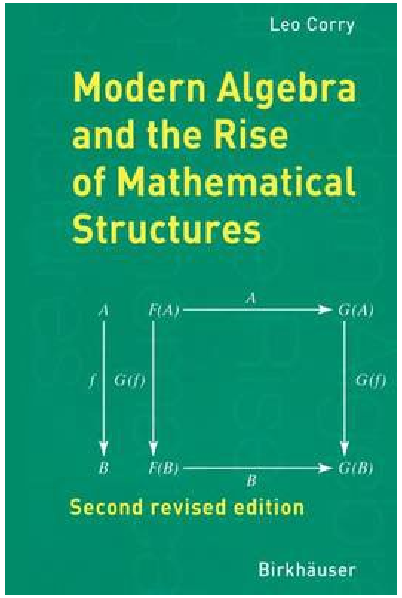
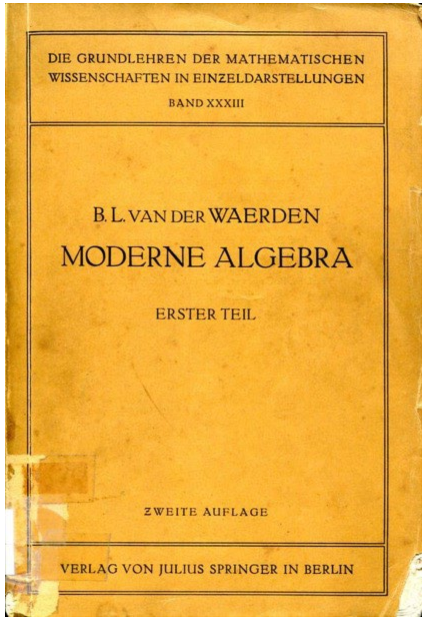
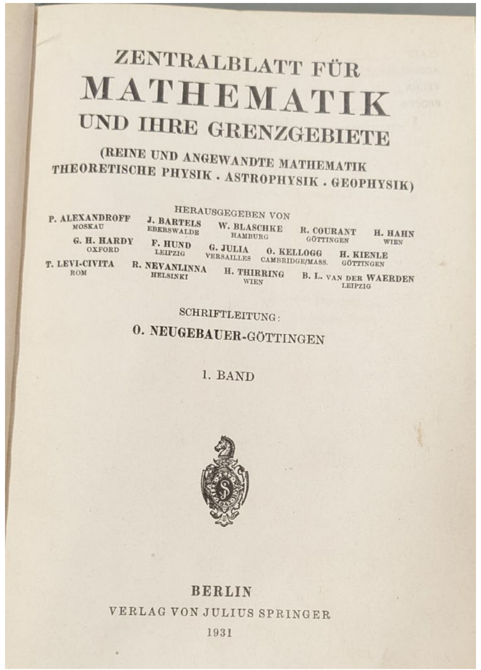
Abstract Relations







Abstract Relations

- Communicating mathematics is hard.
  - *Tangent:* origins of blackboard in mathematics?

Abstract Relations

- Communicating mathematics is hard.
  - *Stabilized* (written up) media circulate abstractions that are mobile but incomprehensible.
  - *Situated* (read down) media mobilize abstractions as workable but contextually constrained productions.

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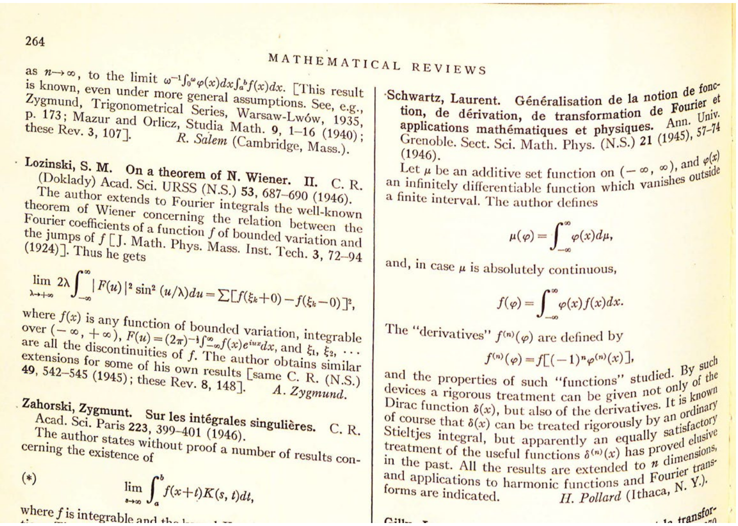
- Communicating mathematics is hard.
  - *Stabilized* (written up) media circulate abstractions that are mobile but incomprehensible.
  - *Situated* (read down) media mobilize abstractions as workable but contextually constrained productions.
- The dynamic of local understanding and distant circulation makes research communities hard to sustain over long distances. (As you know at *Topos*!)

Abstract Relations

- Communicating mathematics is hard.
- Abstracts and reviews are:
  - Short, mobile
  - Prioritize interventions over arguments
  - Indexable (methods of search and identification)
  - Indexical (tell you where things are from, where to look next, how things are related)
    - Gain meaning from what they tell you about where things are relative to other things.



264 MATHEMATICAL REVIEWS

as $n \to \infty$, to the limit $\omega^{-1}\int_0^\omega \varphi(x)dx\int_a^b f(x)dx$. [This result is known, even under more general assumptions. See, e.g., Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 173; Mazur and Orlicz, Studia Math. **9**, 1–16 (1940); these Rev. **3**, 107]. *R. Salem* (Cambridge, Mass.).

Lozinski, S. M. On a theorem of N. Wiener. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 687–690 (1946).

The author extends to Fourier integrals the well-known theorem of Wiener concerning the relation between the Fourier coefficients of a function f of bounded variation and the jumps of f [J. Math. Phys. Mass. Inst. Tech. **3**, 72–94 (1924)]. Thus he gets

$$\lim_{\lambda \to +\infty} 2\lambda \int_{-\infty}^{\infty} |F(u)|^2 \sin^2(u/\lambda)du = \sum[f(\xi_k+0)-f(\xi_k-0)]^2,$$

where $f(x)$ is any function of bounded variation, integrable over $(-\infty, +\infty)$, $F(u)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} f(x)e^{iux}dx$, and $\xi_1, \xi_2, \cdots$ are all the discontinuities of f. The author obtains similar extensions for some of his own results [same C. R. (N.S.) **49**, 542–545 (1945); these Rev. **8**, 148]. *A. Zygmund.*

Zahorski, Zygmunt. Sur les intégrales singulières. C. R. Acad. Sci. Paris **223**, 399–401 (1946).

The author states without proof a number of results concerning the existence of

$$(*) \qquad \lim_{s \to \infty} \int_a^b f(x+t)K(s, t)dt,$$

where f is integrable and the ...

Schwartz, Laurent. Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques. Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946).

Let μ be an additive set function on $(-\infty, \infty)$, and $\varphi(x)$ an infinitely differentiable function which vanishes outside a finite interval. The author defines

$$\mu(\varphi) = \int_{-\infty}^{\infty} \varphi(x)d\mu,$$

and, in case μ is absolutely continuous,

$$f(\varphi) = \int_{-\infty}^{\infty} \varphi(x)f(x)dx.$$

The "derivatives" $f^{(n)}(\varphi)$ are defined by

$$f^{(n)}(\varphi) = f[(-1)^n \varphi^{(n)}(x)],$$

and the properties of such "functions" studied. By such devices a rigorous treatment can be given not only of the Dirac function $\delta(x)$, but also of the derivatives. It is known of course that $\delta(x)$ can be treated rigorously by an ordinary Stieltjes integral, but apparently an equally satisfactory treatment of the useful functions $\delta^{(n)}(x)$ has proved elusive in the past. All the results are extended to n dimensions, and applications to harmonic functions and Fourier transforms are indicated. *H. Pollard* (Ithaca, N. Y.).

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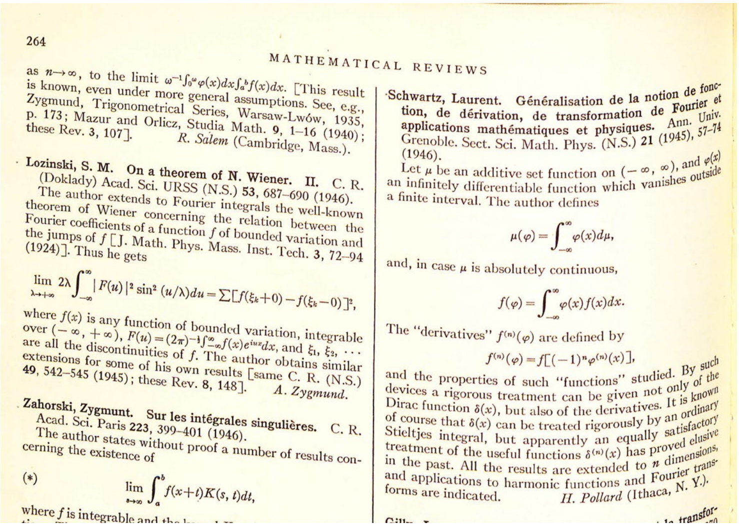
Mathematical abstracts <-> Mathematical abstraction



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MATHEMATICAL REVIEWS

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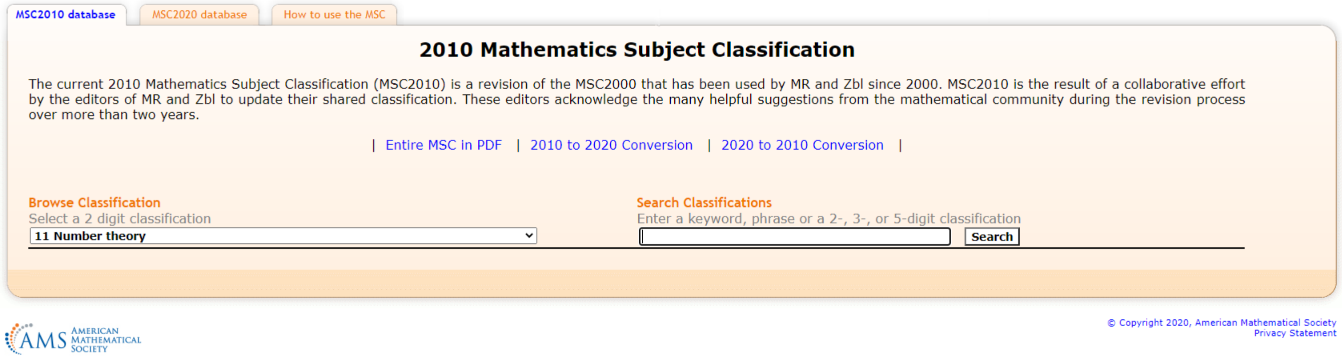
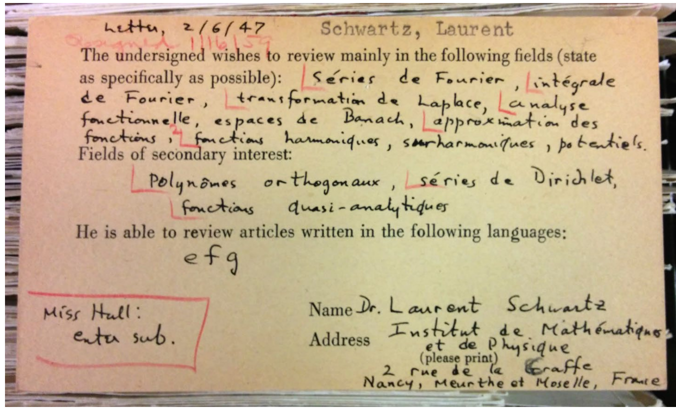
Categories organize and connect people and places

- Names, addresses, languages, fields of study
- Global infrastructure
  - UNESCO support for obtaining reviews in, e.g., Baghdad, Buenos Aires.
  - Reviews support requests for access to other literature and shared sense of a *Weltliteratur* of mathematics.
- Reviews *actively* create conceptual and personal connections.
- Classification systems are social and relational.





Laurent Schwartz reviewer card, Mathematical Reviews headquarters, Ann Arbor

Categories and unification

- Unification is a modern project, reflected in part through anxiety about disintegration
  - E.g. 1912, ‘The Science of Mathematics is now so wide and is already so much specialised that it may be doubted whether there exists to-day any man fully competent to understand mathematical research in all its many diverse branches.’
- Conceptual organisation is disciplinary organisation
  - E.g. 1940, 'Mathematical Physics contains those and only those reviews which are unintelligible.’ 1945 letter arguing against Math. Phys. as a classification.
- Interaction between mathematical and bibliographic thinking
  - E.g. 1939, 'Glad to hear Math. Rev. is making such good progress. I hope their subject classification is as near an ultra-filter as possible.’

Categories and unification: Bourbaki

THE ARCHITECTURE OF MATHEMATICS*

NICHOLAS BOURBAKI†

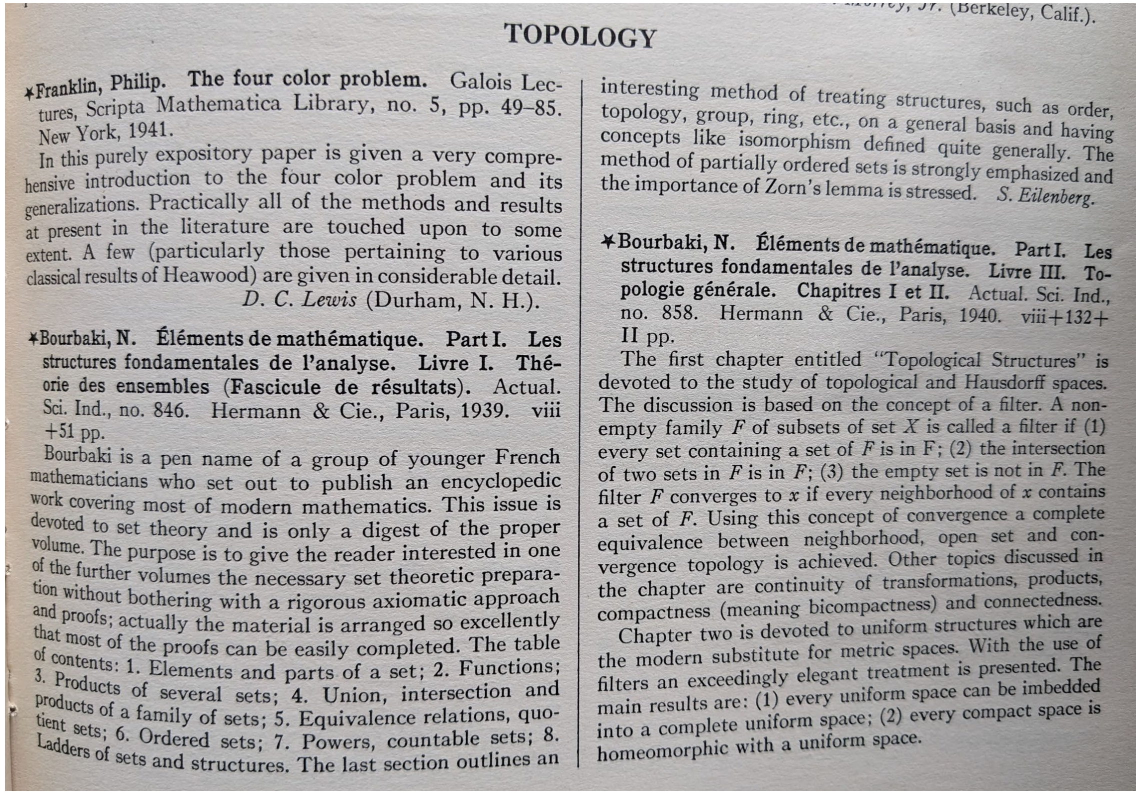
1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first

Besse-en-Chandesse
1935

Scientific American
1957

Categories and unification

- MacLane and Eilenberg’s category theory emerges in part from their engagement with Bourbaki’s project and approach.



Jr. (Berkeley, Calif.).

TOPOLOGY

*Franklin, Philip. **The four color problem.** Galois Lectures, Scripta Mathematica Library, no. 5, pp. 49–85. New York, 1941.

In this purely expository paper is given a very comprehensive introduction to the four color problem and its generalizations. Practically all of the methods and results at present in the literature are touched upon to some extent. A few (particularly those pertaining to various classical results of Heawood) are given in considerable detail. *D. C. Lewis* (Durham, N. H.).

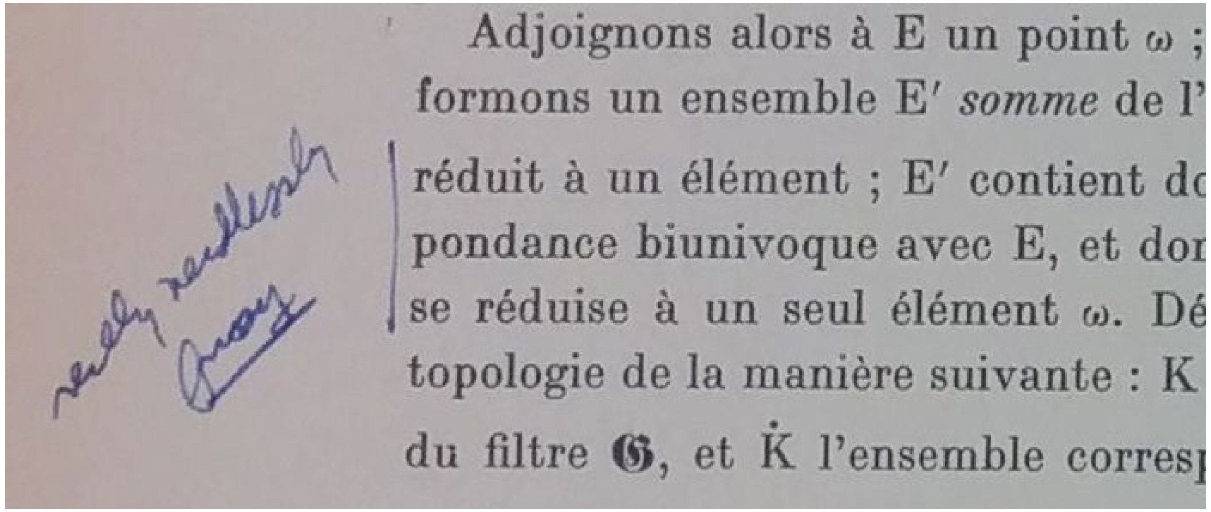
*Bourbaki, N. **Éléments de mathématique. Part I. Les structures fondamentales de l’analyse. Livre I. Théorie des ensembles (Fascicule de résultats).** Actual. Sci. Ind., no. 846. Hermann & Cie., Paris, 1939. viii +51 pp.

Bourbaki is a pen name of a group of younger French mathematicians who set out to publish an encyclopedic work covering most of modern mathematics. This issue is devoted to set theory and is only a digest of the proper volume. The purpose is to give the reader interested in one of the further volumes the necessary set theoretic preparation without bothering with a rigorous axiomatic approach and proofs; actually the material is arranged so excellently that most of the proofs can be easily completed. The table of contents: 1. Elements and parts of a set; 2. Functions; 3. Products of several sets; 4. Union, intersection and products of a family of sets; 5. Equivalence relations, quotient sets; 6. Ordered sets; 7. Powers, countable sets; 8. Ladders of sets and structures. The last section outlines an interesting method of treating structures, such as order, topology, group, ring, etc., on a general basis and having concepts like isomorphism defined quite generally. The method of partially ordered sets is strongly emphasized and the importance of Zorn’s lemma is stressed. *S. Eilenberg.*

*Bourbaki, N. **Éléments de mathématique. Part I. Les structures fondamentales de l’analyse. Livre III. Topologie générale. Chapitres I et II.** Actual. Sci. Ind., no. 858. Hermann & Cie., Paris, 1940. viii+132+II pp.

The first chapter entitled “Topological Structures” is devoted to the study of topological and Hausdorff spaces. The discussion is based on the concept of a filter. A non-empty family F of subsets of set X is called a filter if (1) every set containing a set of F is in F; (2) the intersection of two sets in F is in F; (3) the empty set is not in F. The filter F converges to x if every neighborhood of x contains a set of F. Using this concept of convergence a complete equivalence between neighborhood, open set and convergence topology is achieved. Other topics discussed in the chapter are continuity of transformations, products, compactness (meaning bicompactness) and connectedness. Chapter two is devoted to uniform structures which are the modern substitute for metric spaces. With the use of filters an exceedingly elegant treatment is presented. The main results are: (1) every uniform space can be imbedded into a complete uniform space; (2) every compact space is homeomorphic with a uniform space.

Samuel Eilenberg review of Bourbaki *Set Theory,* Mathematical Reviews, 1942.

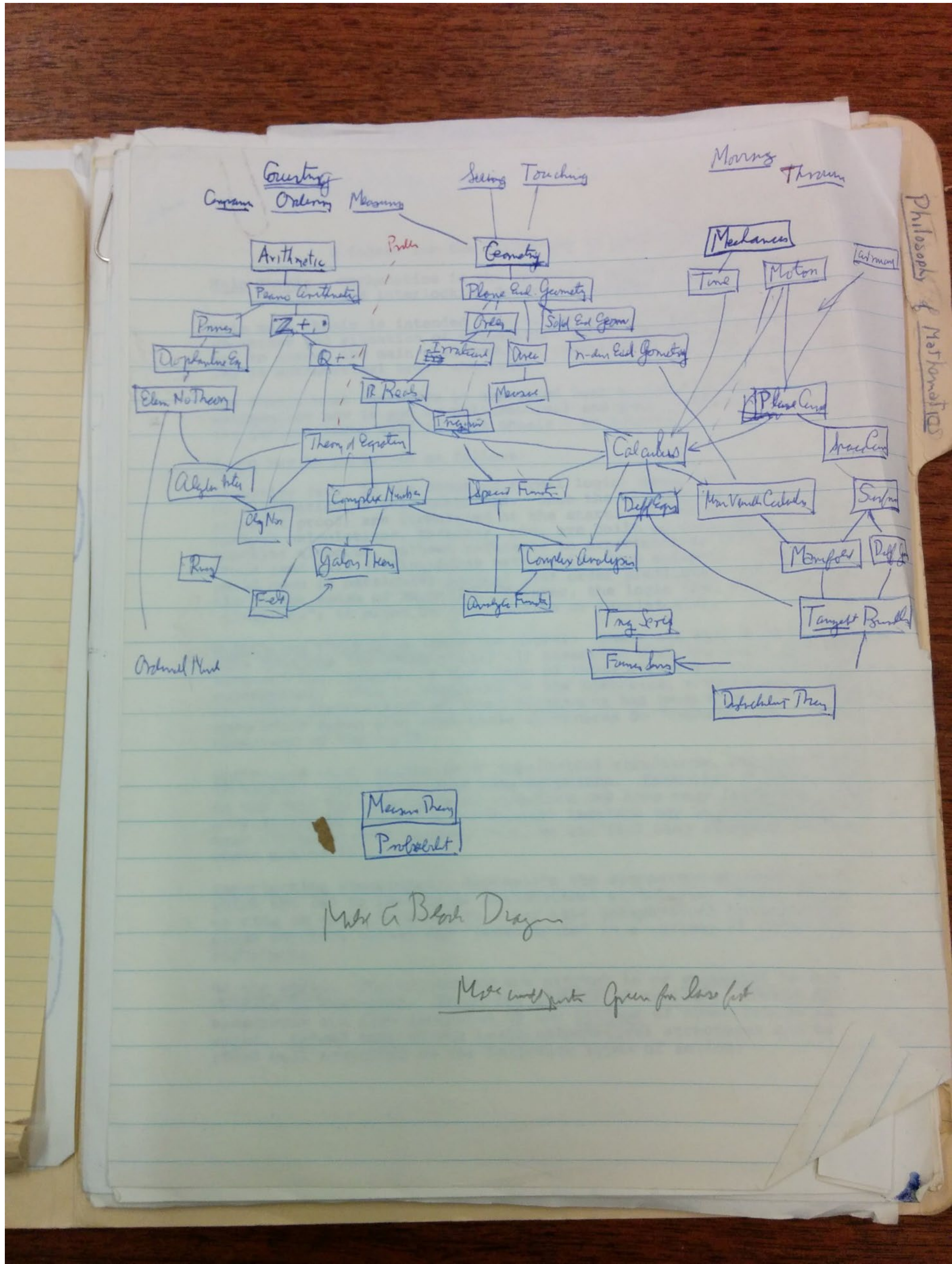


Adjoignons alors à E un point ω ;
formons un ensemble E′ *somme* de l’
réduit à un élément ; E′ contient do
pondance biunivoque avec E, et dor
se réduise à un seul élément ω. Dé
topologie de la manière suivante : K
du filtre 𝔊, et K̇ l’ensemble corres

Saunders MacLane annotation of Bourbaki, *Topologie Générale,* p. 66.
Papers of Saunders MacLane, University of Chicago Library.

Categories and unification

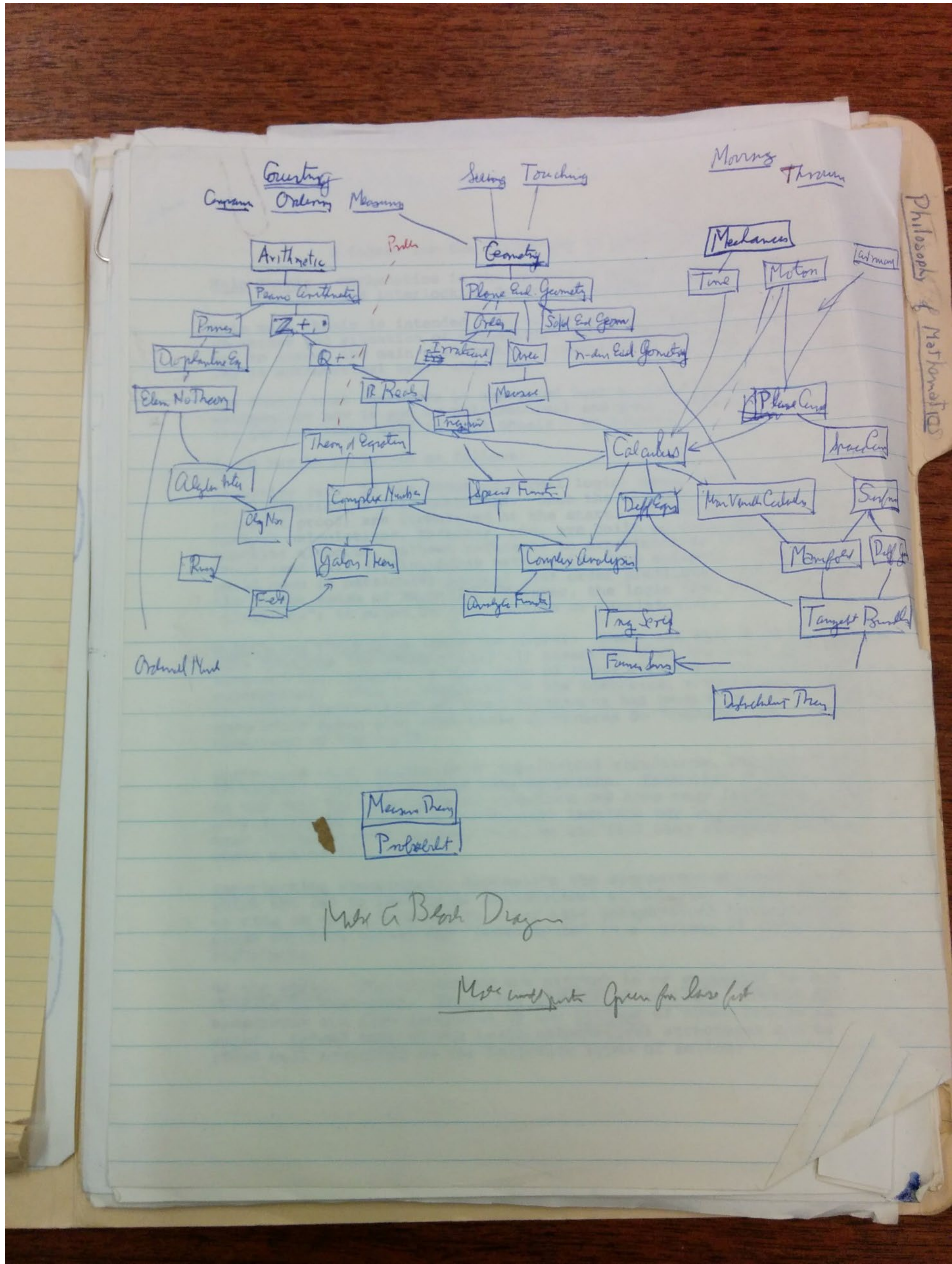
- MacLane's *way of thinking* with diagrams and arrows.
    - Some evidence this was a draft of diagrams used in teaching or lecturing c. 1979.



Saunders MacLane, "Philosophy of Mathematics" undated diagram
Papers of Saunders MacLane, University of Chicago Library.

Categories and unification

- MacLane's *way of thinking* with diagrams and arrows.
  - Some evidence this was a draft of diagrams used in teaching or lecturing c. 1979.
- Structures, relations, *sensations, matter.*
  - Scrap paper and annotations as *situated* media representing the dynamic aspects of categorical thinking.



Saunders MacLane, "Philosophy of Mathematics" undated diagram
Papers of Saunders MacLane, University of Chicago Library.

Two inversions of perspective

- Social Structure: arrangements of social institutions that characterize how members of a society relate to each other.

Two inversions of perspective

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- Seeing and creating structural worlds were responses to changing scales and goals of mathematical research, with conjoined mathematical/organizational/personal effects.

How Categories Come to Matter

TOPOS INSTITUTE

To advance the sciences of connection and integration, we draw on rich mathematical frameworks for modelling relationships, including *category theory*, *topos theory*, and *type theory*, as well as a long tradition of practical construction of tools in *programming languages*, *machine intelligence*, and *ubiquitous computing*.

https://topos.institute/work

- Recent mathematical frameworks for modelling (mathematical) relationships *grew from* and *thrived within* modern mathematical practices of sustaining (human and institutional) relationships.
- The materiality of abstract relationships permits historical and sociological analysis of the social, institutional, geopolitical and other dimensions of abstraction.

How Categories Come to Matter

On the History and Sociology of Categories in Modern Mathematics

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