How to Interpret Cotorsion?

> Mathematical models as we encounter them in practice may be expressed by ordinary or partial differential equations, they may involve the language of graphs or lattice diagrams, or require the notion of transfer function or a formal language.
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> "Paradigms and puzzles in the theory of dynamical systems", by Jan C. Willems

① A (linear) control system is an underdetermined system of (linear) differential equations, i.e., systems some of whose variables are free (i.e, can be chosen arbitrarily.)

Remark. In real life all control systems are nonlinear. To use linear methods, systems have to be linearized first.

The free functions form the input of the system. Manipulating the input one tries to guarantee a desired behavior of the system. This requires an output, i.e., output variables.

Thus we have a system Σ :

$$(1) \qquad \begin{aligned} x'(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

Here:
- x is the state of the system
- u is the input
- y is the output, and

A, B, C, D are matrices with constant, or polynomial, or analytic coefficients. The base field is $\mathbb{R}$ or $\mathbb{C}$.

Symbolically, the system is represented by a formal diagram

$$\xrightarrow{\ u\ } \boxed{\quad x \quad} \xrightarrow{\ y\ }$$

Associated with Σ is the dual system $\bar{\Sigma}$:

$$(2) \qquad \begin{aligned} x'(t) &= A^T x(t) + C^T u(t) \\ y(t) &= B^T x(t) + D^T u(t) \end{aligned}$$

where T stands for the transpose

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(2) Systems could be multi-dimensional. Thus we look at partial derivatives $\frac{\partial f}{\partial t_i}$ and, after decoupling these symbols, at the differential operators $\frac{\partial}{\partial t_i}$. Together with a choice of (constant or functional) coefficients, they form a ring, D, of differential operators.

A multidimensional system rewrites as

$$(*) \qquad A\bar{x} = 0,$$

where A is a finite matrix with entries in D and $\bar{x}$ is a column vector with entries in a suitable class of functions.

Thinking of $\bar{x}$ as unknown elements in a module, we have a system of linear equations in that module, and the solutions of this system are the solutions of the original system of differential equations.

At this point it is convenient to look at the solutions of $(*)$ not just in the chosen module but in all D-modules. This yields a "solution space" functor (with values in abelian groups).

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③ Bringing in functors.

Proposition. The solution space functor

$$S : \text{Mod-}D \longrightarrow Ab$$

is representable. The representing object M is defined by the exact sequence

$$F_1 \xrightarrow{A^T} F_0 \longrightarrow M \longrightarrow 0.$$

In other words, $S(L) \simeq \mathrm{Hom}_D(M, L)$ for any D-module L. This is known as the Malgrange isomorphism.

4 Back to control systems.

Two important features of a control system are controllability and observability.

Definition. The system Σ is controllable if it can be taken from any initial state to any finite state by choosing an appropriate input.

Definition. The system Σ is observable if the state of the system can be recovered from the output.

Theorem The dual system $\bar{\Sigma}$ is observable if and only if Σ is controllable. $\boxtimes$

The controllability of Σ can be expressed in terms of the representing module M.

Proposition. Σ is controllable if and only if M has no torsion. $\boxtimes$

To speak of the torsion of a module one has to make some assumptions on the ring. In the case when the ring is the ring of integers or, more generally, a commutative domain the torsion submodule of a module is just the totality of all its elements that can be annihilated by nonzero elements of the ring.

<u>Example.</u> Let the ring be $\mathbb{Z}$. It can be viewed as a module over itself. Its torsion submodule consists of a single element, 0. On the other hand, the torsion submodule of $\mathbb{Z}/4$ (integers modulo 4) is the entire $\mathbb{Z}/4$ since any element of it is annihilated by $4 \in \mathbb{Z}$.

<u>Our goals now are</u>:

(I) <u>To introduce a definition of torsion that would work for any module over any ring.</u>

(II) <u>To propose a conjectural algebraic interpretation of observability.</u>

(III) <u>To propose a functorial framework for the duality between observability and controllability.</u>

(5) Redefining torsion. (M-R, 2020)

The classical torsion was first observed and named by H. Poincaré in a topological context around 1900. A formal algebraic definition was only given in the 1920s.

To generalize the classical torsion for modules over commutative domains we return momentarily to modules over $\mathbb{Z}$, i.e., abelian groups. Then the torsion submodule of an abelian group A can also be defined as the kernel of the localization map. To wit, embed $\mathbb{Z}$ in the rational numbers

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Q}$$

and tensor this sequence with A. The kernel of the resulting map is precisely $T(A)$, the torsion submodule of A:

$$0 \longrightarrow T(A) \longrightarrow A \otimes \mathbb{Z} \xrightarrow{A \otimes \iota} A \otimes \mathbb{Q}$$

Exactly the same procedure works over any commutative domain, just replace $\mathbb{Q}$ with the field of fractions of the domain.

Now notice that $\mathbb{Z} \xrightarrow{\iota} \mathbb{Q}$ is the injective envelope of $\mathbb{Z}$. In fact, for our purposes, it suffices to notice that $\mathbb{Q}$ is injective, and more generally, the same is true for the field of fractions of any commutative domain.

This motivates a general definition of torsion that works for arbitrary modules over arbitrary rings.

Definition Let Λ be an arbitrary associative ring with identity and A_Λ a right Λ-module. To define the torsion submodule $s(A) \subseteq A$, do the following:

ⓐ View Λ as a left module over itself: ${}_\Lambda\Lambda$. Embed it into an injective I

$$0 \longrightarrow {}_\Lambda\Lambda \xrightarrow{\ \iota\ } I$$

ⓑ Tensor this sequence with A:

$$\begin{array}{ccc} A \otimes_\Lambda \Lambda & \xrightarrow{A\otimes\iota} & A \otimes I \\ \wr\wr & & \\ A & & \end{array}$$

ⓒ Take the kernel of $A \otimes \iota$.

Thus we obtain a defining exact sequence

$$0 \longrightarrow s(A) \longrightarrow A \xrightarrow{A\otimes\iota} A \otimes I.$$

⑥ <u>Functors come into play.</u>

Let $F : \Lambda\text{-Mod} \longrightarrow Ab$ be an additive functor from left Λ-modules to abelian groups. Associated with F are its right-derived functors R^iF, $i \geq 0$. Of special interest to us is the zeroth right-derived functor and the canonical natural transformation

$$\rho_F : F \longrightarrow R^0F.$$

(Recall that ρ_F is an isomorphism if and only if F is left-exact and that ρ_F is an isomorphism on injectives.)

<u>DEFINITION.</u> <u>The kernel of ρ_F, denoted by $\overline{F}$, is called the injective stabilization of F. Thus we have a defining sequence</u>

$$0 \longrightarrow \overline{F} \longrightarrow F \xrightarrow{\rho_F} \overline{F}$$

<u>Remark</u> <u>F is the largest subfunctor of F vanishing on injectives.</u>

The components of $\overline{F}$ admit a simple description. To compute $F(B)$ on a left Λ-module B do the following:

(a) Embed B in an injective I:
$$0 \longrightarrow B \xrightarrow{\imath} I$$

(b) Apply F:
$$F(B) \xrightarrow{F(\imath)} F(I)$$

(c) Take the kernel : $\bar{F}(B) = \operatorname{Ker} F(\imath)$.

Thus we have a defining sequence
$$0 \longrightarrow \bar{F}(B) \longrightarrow F(B) \xrightarrow{F(\imath)} F(I).$$

Specializing to the case $\bar{F} = A \otimes_\Lambda -$ and $B := {}_\Lambda\Lambda$ we have proved

<u>Theorem</u>

(MR-2020) $\quad s(A) = (A \bar{\otimes} -)({}_\Lambda\Lambda)$. $\boxtimes$

We have achieved the first of the three goals mentioned above – we have a general definition of torsion.

(7) <u>Introducing cotorsion</u> (M-R, 2020)

Now we define the cotorsion of an arbitrary module over an arbitrary ring.

<u>Remark</u>. <u>Unlike torsion, the notion of cotorsion does not have a historical prototype.</u>

This will be done by "dualizing" the notion of torsion. This is possible because torsion was defined functorially.

First we replace $F = A \otimes -$ by a "dual" functor $F = (-, C)$. Instead of taking the injective stabilization $A \overline{\otimes} -$, take the projective stabilization $(\underline{-, C})$.

(The projective stabilization $\underline{F}$ of an additive functor F is defined as the cokernel of the canonical natural transformain

$$L_0 F \xrightarrow{\lambda_F} F \longrightarrow \underline{F} \longrightarrow 0 \,).$$

And, as the last step, instead of $F(\Lambda)$, take $\underline{F}(\Lambda)$.

<u>Definition</u> <u>The cotorsion of a module C is defined by</u>:

$$q(C) := (\underline{-, C})(\Lambda).$$

Because $(-, C)$ is a contravariant functor the right-hand side of the defining formula is related to injectives, not projectives!

<u>Fact</u> $(\underline{-, C})(\Lambda) = (\overline{\Lambda, C})$, where the latter denotes (for historial reasons) Hom modulo injectives, i.e. $(\overline{\Lambda, C}) = (\Lambda, C)/\mathcal{I}(\Lambda, C)$, where $\mathcal{I}(\Lambda, C)$ denotes the subgroup of all maps $\Lambda \longrightarrow C$ factoring through injectives.

Fact. Since $(\Lambda, C) \cong C$, $(\Lambda,C)/\mathcal{I}(\Lambda,C)$ is isomorphic to C modulo the trace of of injective modules in C, i.e. we mod out the submodule of C generated by the images of all homomorphisms from all injectives into C.

In particular, q is a quotient of the identity functor.

We have thus achieved our second goal: we have a definition of cotorsion that works for arbitrary modules over arbitrary rings.

Observation:

We now have short exact sequences of endofunctors

$$0 \longrightarrow s \longrightarrow \mathbb{1} \longrightarrow s^{-1} \rightarrow 0 \quad \text{and}$$

$$0 \longrightarrow q^{-1} \longrightarrow \mathbb{1} \longrightarrow q \rightarrow 0$$

It can be shown that s is a radical, i.e. $ss^{-1} = 0$, and that q is a coradical (i.e. a radical on the opposite category), i.e. $qq^{-1} = 0$.

(8) The Auslander – Gruson – Jensen duality

Recall that a functor $F : \text{Mod-}\Lambda \longrightarrow Ab$ is said to be finitely presented if it is defined by an exact sequence

$$(B,-) \xrightarrow{(f,-)} (A,-) \longrightarrow F \longrightarrow 0 .$$

The totality of all such functors is an abelian category. The same is true for functors finitely presented by finitely prsented modules and viewed as functors on the category $\text{mod}\,\Lambda$ of finitely presented modules.

(A module is said to be finitely presented if it is the cokernel of a homomorphism between finitely generated free modules.) We denote this functor category by $fp\,(\text{mod}(\Lambda), Ab)$.

The Auslander – Gruson – Jensen duality, D, relates $fp\,(\text{mod}(\Lambda^{op}), Ab$ and $fp\,(\text{mod}(\Lambda), Ab)$:

$$\begin{tikzcd} fp\,(\text{mod}\,(\Lambda^{op}), Ab) \arrow[r, bend left, "D"] & fp\,(\text{mod}(\Lambda), Ab) \arrow[l, bend left, "D"] \end{tikzcd}$$

Given a functor F, the value of DF on B is given by

$$DF(B) := \mathcal{N}at(F, B \otimes -)$$

A similar formula defines DF on natural transformations:

$$DF(\alpha) := \mathcal{N}at(\alpha, B \otimes -)$$

Properties of D

1) D is contravariant.

2) D interchanges Hom's and tensor products.

3) D is exact.

The next result shows that torsion is completely determined by cotorsion (on the other side!).

Theorem. (MR-2020) $Dq = s$. $\boxtimes$

Returning now to the two short exact sequences of endofunctors, we have $Dq = s$ by the theorem just stated. $D(\mathbb{1}) = \mathbb{1}$ is always true. Therefore, by the exactness of D, we have

Corollary $\boxed{Dq^{-1} = s^{-1}.}$

Symbolically,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, phantom, "D"] & q^{-1} \arrow[r] \arrow[drr, dashed, leftrightarrow] & 1 \arrow[r] \arrow[d] & q \arrow[r] \arrow[dll] & 0 \\
0 \arrow[r] & S \arrow[r] & 1 \arrow[r] & S^{-1} \arrow[r] & 0
\end{tikzcd}$$

Going back to control systems, the torsion submodule of the module M associated with a system "corresponds" to the part of the system that cannot be controlled. That part is known as the autonomy of the system.

Assuming now that our torsion S could be viewed as the "autonomy functor", it is natural to suggest that S^{-1} should be the "controllable part functor".

The duality between the observability and controllability leads us to a conjecture that the above corollary provides a functorial framework for such a duality. In particular, q^{-1} should be viewed as a functorial description of the notion of observability.

9 Back to modules.

The preceding corollary is a duality on functors which must be evaluated on the same module on the left and on the right. Is there a duality that "desends" to modules? The answer is yes.

Assume that Λ is an algebra over a commutative ring R. This R could be $\mathbb{Z}$, so this assumption does not impose any restrictions. Let J be an injective R-module and $D_J := \mathrm{Hom}_R(-, J)$.

Theorem (MR-2019). $\boxed{D_J(A \bar{\otimes} B) \simeq \overline{(B, D_J(A))}}$ $\boxtimes$

Here $(A \bar{\otimes} B)$ stands for $(A \bar{\otimes} -)(B)$.

Now specialize to the case $B := {}_\Lambda\Lambda$.

Corollary $D_J(s(A)) \simeq q(D_J(A))$ and, similar to the above, $D_J(s^{-1}(A)) \simeq q^{-1}(D_J(A))$ $\boxtimes$

Conjecture. For a suitable choice of J, $D_J(A)$ should represent the system dual to the system represented by A, and $q^{-1}(D_J(A))$ should be the observable part of the dual system.

References

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