Geometric principles of data visualization

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Thus, human psychology may not directly help us and we need mathematics!

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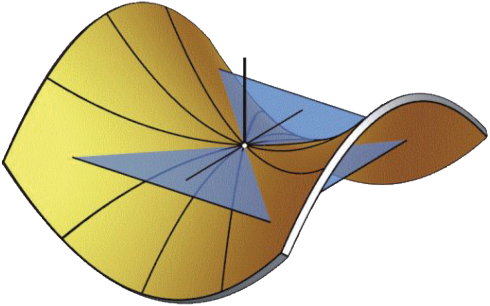
For purposes of visualization, it would even be best if those representations are 2D or 3D.

Methods of Machine Learning

We assume that the data come with some metric structure, that is, distances between data points. These could be Euclidean distances (when the data are presented in some Euclidean space, where the axes record the values of particular quantitative features) or abstract inner distances (dissimilarities). Machine learning then extracts features of the data from these metric relations.

Some methods of Machine Learning

1. Manifold learning
   Data given in some high-dimensional (Euclidean) space, but are assumed to lie on or be concentrated near some low-dimensional smooth submanifold, which may stretch into many ambient directions. This manifold should be recovered by sampling the data.



Graphics courtesy of G. Gläser and K. Polthier

Some methods of Machine Learning

1. Manifold learning
2. Network analysis
   Represent data as graphs, by connecting data points that are sufficiently close or similar or have a particular relation. Use concepts from graph theory to identify particular features.[1]

[1] E.g. M.Eidi, A.Farzam, W.Leal, A.Samal, JJ, *Theory Biosc.*, 2020

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3. Topological data analysis (persistent homology)
   For each radius $r > 0$, turn the intersection pattern of the balls $B(x, r)$ around the data points into a simplicial complex and compute its topological invariants (homology classes). Identify those homology classes that persist for a large range of radii, as encoding important morphological features of the data at their persistence scale.

Machine Learning and Mathematics

1. In manifold learning, a graph is constructed by joining sufficiently close sample points. With more and more samples, eigenvalues/eigenfunctions of graph Laplacian approximate those of Laplace-Beltramie operator of manifold, to recover it.

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[2] JJ, R.Mulas, D.Zhang, *Spectra of discrete structures*, Cambridge Texts in Advanced Math., to appear

[3] E.g. M.Eidi, JJ, *Sci.Rep.*, 2020; JJ, F.Münch, A.Samal, E.Saucan, *Discrete curvatures and their applications*, monograph, in preparation

[4] P.Joharinad, JJ, *Geometric methods of data analysis*, Mathematics of Data, Springer

Machine Learning and Mathematics

1. In manifold learning, a graph is constructed by joining sufficiently close sample points. With more and more samples, eigenvalues/eigenfunctions of graph Laplacian approximate those of Laplace-Beltramie operator of manifold, to recover it.
2. Network analysis can use eigenvalues and eigenfunctions of graphs and hypergraphs[2] and of curvature statistics,[3] as some foundations of machine learning and data analysis.
3. Intersection patterns of balls in TDA can be related to curvature notions for metric spaces.[4]

[2] JJ, R.Mulas, D.Zhang, *Spectra of discrete structures*, Cambridge Texts in Advanced Math., to appear

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I shall present a method, **IsUMap**, developed in

L.Barth, H.Fahimi, P.Joharinad, J.J., J.Keck, *Data visualization with category theory and geometry*, Math of Data, Springer, to appear

that builds upon and improves the popular **UMAP**.[5]

[5] McInnes, Healy and Melville, *UMAP: Uniform Manifold Approximation and Projection for Dimension Reduction*, arxiv, 2018

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- and the data need not be uniformly distributed on that manifold
- and the data should ultimately be visualized in 2 or 3D, preserving the qualitative aspects, like clusters, as well as possible

Existing methods

1. **Isomap:** Tenenbaum, Silve and Langford, *A global geometric framework for nonlinear dimensionality reduction*, Science, 2000
2. **t-SNE:** Hinton and Roweis, *Stochastic neighbor embedding*, Adv. Neur. Inf. Proc. Sys, 2002
3. **UMAP:** McInnes, Healy and Melville, *UMAP: Uniform Manifold Approximation and Projection for Dimension Reduction*, arxiv, 2018

General idea:

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Approach: Preprocess and modify the graphs to amplify local features, like clusters, before projecting.

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2. Develop a canonical scheme for merging these star graphs, with tools from category theory
3. Project the resulting space onto 2D with minimal distortion

Riemannian geometry

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$$\langle V, W \rangle = \sum_{i,j=1,\ldots,n} g_{ij}(x) v^i w^j$$

for $V = \sum_i v^i \frac{\partial}{\partial x^i}, W = \sum_i w^j \frac{\partial}{\partial x^j}$ at a point x.

[6] J.J., *Bernhard Riemann, On the hypotheses...*, 2nd ed. Birkhäuser, 2025;

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Bernhard Riemann (1826-1866) in 1854 introduced the abstract concept of a manifold of arbitrary dimension n equipped with a metric tensor $(g_{ij})_{i,j=1,\dots,n}$ with which one can compute a scalar product between tangent vectors at a point. Such a Riemannian manifold need not be realized in any ambient Euclidean space, but is intrinsically determined by its metric tensor.

$$\langle V, W \rangle = \sum_{i,j=1,\dots,n} g_{ij}(x) v^i w^j \tag{1}$$

for $V = \sum_i v^i \frac{\partial}{\partial x^i}, W = \sum_i w^j \frac{\partial}{\partial x^j}$ at a point x.
But this is represented in local coordinates. In other coordinates, the tangent vectors V, W and the metric tensor (g_{ij}) look different, but they transform in such a way that the scalar quantity (1) remains invariant.
This is the principle of *covariance* that is fundamental for Einstein's *theory of general relativity*. It tells us that we should look for intrinsic properties that do not depend on the representation.

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Riemann[6] solved the problem of determining a complete set of local invariants for such a metric tensor. From the *curvature tensor* with components

$$R^k_{\ell ij} = \frac{\partial \Gamma^k_{j\ell}}{\partial x^i} - \frac{\partial \Gamma^k_{i\ell}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{j\ell} - \Gamma^k_{jm}\Gamma^m_{i\ell} \qquad (1)$$

with $\Gamma^i_{jk} = \frac{1}{2} g^{i\ell}(\frac{\partial}{\partial x^k} g_{j\ell} + \frac{\partial}{\partial x^k} g_{k\ell} - \frac{\partial}{\partial x^\ell} g_{jk})$ (summation signs over double indices omitted; $g^{i\ell}$ is the inverse of the metric tensor), he extracts the **sectional curvatures**.

[6] J.J., *Bernhard Riemann, On the hypotheses...*, 2nd ed. Birkhäuser, 2025; J.J. *Riem.Geom.& Geom.Anal.*, 7th ed. Springer, 2017

Somewhat surprisingly, for the moment, we do not need curvatures, only the concept of a Riemannian metric.

In fact, Riemann introduced another important device, that of *normal coordinates*. Essentially, given a point p in a Riemannian manifold M, for other points q in the vicinity of p, you record the distance $d(p, q)$ and the angles that the shortest geodesic from p to q makes with some reference direction at p. Like euclidean polar coordinates. These coordinates exhibit what the manifold locally looks like from the perspective of p.

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This does not work globally, because in general the shortest geodesic from p to a more distant point in M need not be unique.

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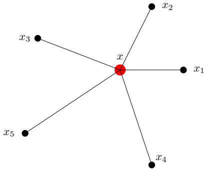
Thus, in a discrete metric space, for a point x, we take its k nearest neighbors $x_1, \ldots, x_k$, ranked according to the distance from p and consider the star graph $S(x)$ with x as its central vertex and edges of length $d(x, x_j)$ to the x_j.



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However, for computational reasons, that metric needs to be modified.

Equip the star graphs with a (generalized) metric

$$\begin{aligned} d_x(x, x_j) &= \frac{d(x, x_j) - \rho_x}{\sigma_x} \quad \text{for } j = 1, \ldots k \\ d_x(y, y) &= 0 \quad \text{for all } y \\ d_x(x_j, x_\ell) &= \frac{1}{\sqrt{2}}(d_x(x_j, x) + d_x(x, x_\ell)) \quad \text{for neighbors } x_j, x_\ell \text{ of } x \\ d_x(y, z) &= \infty \quad \text{in all other cases} \end{aligned} \tag{2}$$

where $d(.,.)$ is the original distance function on the data set.

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In UMAP, one puts the distance $= \infty$ between different neighbors of the center x, thereby violating the triangle inequality.

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\end{aligned}
$$

The factor $\frac{1}{\sqrt{2}}$ mimicks the Euclidean distance of the ends of orthogonal vectors. We could also omit it.
The choice of ρ_x and σ_x will be explained in a moment.

Definition (David Spivak)

An *uber-metric space* (X, d) is a set X equipped with a map $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

1. $d(x, y) \geq 0$, and $d(x, x) = 0$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The category of uber-metric spaces $\mathbf{UM}$ has as objects uber-metric spaces and as morphisms again non-expansive maps. The category of finite uber-metric spaces is denoted by $\mathbf{FinUM}$.

The third property in the definition of uber-metric space implies that if $d(x, z) = \infty$ in an uber-metric space, then for any y either $d(x, y) = \infty$ or $d(z, y) = \infty$.

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The choice of ρ_x eases the curse of dimension, because randomly picked points in a high dimensional ball tend to concentrate near the outer boundary. For many data sets, however, already $\rho_x = 0$ works well.

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σ_x achieves some normalization. This is important if the data points are non-uniformly distributed on the manifold.

$$d_x(x, x_j) = \frac{d(x, x_j) - d(x, x_1)}{d(x, x_k)} \quad \text{for } j = 1, \ldots k \tag{4}$$

$$d_x(x_j, x_\ell) = \frac{1}{\sqrt{2}}(d_x(x_j, x) + d_i(x, x_\ell)) \text{ for neighbors of } x \tag{5}$$

$$d_x(y, z) = \infty \quad \text{in all other cases.} \tag{6}$$

Thus, by (4), the closest neighbor x_1 of x has distance 0, and the largest neighbor x_k has distance < 1. By (5), the vertices of the star graph satisfy a triangle inequality, as if sitting in independent directions. By (6), vertices not in the star have infinite distances and therefore are not seen from the center.

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[7] K.Menger, *PNAS*, 1942

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Options:

1. Develop a canonical scheme for merging metric spaces
2. Convert them into fuzzy graphs that can be merged in a canonical way with the help of a t-conorm, utilizing constructions from category theory. (Fuzzy graphs are a special case of Spivak's fuzzy simplicial sets. See below)

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I shall develop here the first alternative, but the cognoscenti will see the analogy with Spivak's fuzzy simplicial sets. An essential idea, however, already goes back to Karl Menger.[7]

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Hazy sets

Definition

Topology on $[0,\infty]$: open sets $=$ intervals $(s,\infty]$, indicated simply by s. $i_{ts} : t \to s$ is inclusion $(t,\infty] \subset (s,\infty]$ for $s \leq t$. Category $\mathbf{H}$.

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We call s the *haziness* of $S(s)$ and consider $S(s)$ as the set of haziness at most s.
All $S(s)$ are subsets of $X := S(\infty)$. Set of haziness precisely t

$$S(=t) := S(t) \setminus \bigcup_{s>t} S(s). \tag{7}$$

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A *hazy simplicial set* is a functor $\mathbf{\Delta^{op}} \to \mathbf{Haz}$ where Δ is the simplicial indexing category. Functor category $\mathbf{sHaz}$ of hazy simplicial sets.

A hazy simplicial set is a functor $S : (\mathbf{\Delta} \times \mathbf{H})^{\mathsf{op}} \to \mathbf{Set}$.
$S^n_s := S([n], (s, \infty])$ are the n-simplices in the image of S of haziness at most s.

Lemma

A simplex is at least as hazy as its faces.

Hazyness becomes metric

Yoneda embedding $(\Delta \times \mathbf{H}) \to \mathbf{Set}^{\Delta \times \mathbf{H}^{op}}$ yields functors Δ_s^n with

$$\Delta_s^n(m, t) = \mathrm{Hom}_{\Delta \times \mathbf{H}}((m, t), (n, s)). \tag{8}$$

Such morphisms exist only for $s \leq t$. And so, morphisms $\Delta_s^n \to \Delta_r^\ell$ (obtained by composing morphisms $((m, t) \to (n, s) \to (\ell, r))$ can exist only for $r \leq s$. We consider Δ_s^n as the n-simplices of haziness at most s.

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The *smear functor* $\mathbf{Sm}$ associates to Δ^n_s a geometric n-simplex of diameter s,

$$\mathbf{Sm}_\Delta(\Delta^n_s) := \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x^i \geq 0 \text{ and } \sum_{i=1}^{n+1} x^i = s \right\}. \tag{9}$$

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Using Kan extensions (while a functor need not admit adjoints, its lift to the Yoneda categories of presheaves does), $\mathbf{Sm}$ becomes a functor from $\mathbf{sHaz}$ to $\mathbf{UM}$.

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Yoneda embedding $(\Delta \times \mathbf{H}) \to \mathbf{Set}^{\Delta \times \mathbf{H}^{op}}$ yields functors Δ^n_s with

$$\Delta^n_s(m,t) = \mathrm{Hom}_{\Delta \times \mathbf{H}}((m,t),(n,s)). \tag{8}$$

Such morphisms exist only for $s \leq t$. And so, morphisms $\Delta^n_s \to \Delta^\ell_r$ (obtained by composing morphisms $((m,t) \to (n,s) \to (\ell,r))$ can exist only for $r \leq s$. We consider Δ^n_s as the n-simplices of haziness at most s.

The *smear functor* $\mathbf{Sm}$ associates to Δ^n_s a geometric n-simplex of diameter s,

$$\mathbf{Sm}_\Delta(\Delta^n_s) := \left\{ x \in \mathbb{R}^{n+1} \,\middle|\, x^i \geq 0 \text{ and } \sum_{i=1}^{n+1} x^i = s \right\}. \tag{9}$$

On morphisms, it operates by rescaling by $\frac{s}{t}$ when $s \leq t$.

Image of hazy simplicial set is uber-metric: Simplices have metric (9), compatible with glueing along shared faces. Faces cannot be more hazy than simplex itself, hence diameter not larger, shortest path can't be shortened by going into a higher-dimensional simplex.

From an uber-metric space to a simplicial hazy set

We also want to go in the opposite direction.

Definition

The *tightening functor* is

$$\begin{aligned}\mathbf{Ti} : \mathbf{UM} &\to \mathbf{sHaz}\\ Y &\mapsto \mathbf{Ti}(Y) : (\Delta \times \mathbf{H})^{\mathsf{op}} \to \mathbf{Set}\\ &\qquad (n, s) \mapsto \mathrm{Hom}_{\mathbf{UM}}(\mathbf{Sm}(\Delta^n_s), Y)\ .\end{aligned} \tag{10}$$

Morphisms in $\mathbf{UM}$ are distance non-increasing. Therefore, the larger s, the more morphisms there are into a given uber-metric space Y, because $\mathbf{Sm}(\Delta^n_s)$ is a simplex with diameter s.

Theorem

$\mathbf{Ti}$ *is right adjoint to* $\mathbf{Sm}$, *i.e.*

$$\mathrm{Hom}_{\mathbf{UM}}(\mathbf{Sm}(S), Y) \simeq \mathrm{Hom}_{\mathbf{sHaz}}(S, \mathbf{Ti}(Y))\ . \tag{11}$$

The merger tool

Definition

An *m-scheme* is a function $M : [0,\infty] \times [0,\infty] \to [0,\infty]$ with:

(1) Symmetry : $M(s,t) = M(t,s)$,

(2) Monotonicity : $M(s,t) \leq M(v,w)$ if $s \leq v$ and $t \leq w$,

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Any m-scheme satisfies

$$M(s,0) = M(0,s) = 0 \quad \text{for all } s \ . \tag{12}$$

This follows from $M(0,\infty) = 0$, monotonicity and symmetry.

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Extremal examples:

- $M_{\min}(s,t) = \min(s,t)$
- $M_{\text{ext}}(s,t) = \begin{cases} t & \text{if } s = \infty \\ s & \text{if } t = \infty, \\ 0 & \text{else.} \end{cases}$

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$M_{\min}(s,t) = \min(s,t)$, $M_{\text{ext}}(s,t) = 0$ if $s,t \neq \infty$ and (4).
Let $0 \in V \subset \mathbb{R}^{\geq 0}$,

$$M_V(s,t) := \begin{cases} t & \text{if } s = \infty \\ s & \text{if } t = \infty, \\ \sup_{v\in V}\{\ v \mid v \leq s \text{ and } v \leq t\ \} & \text{else.} \end{cases} \tag{12}$$

For $V = \mathbb{R}^{\geq 0}$, we get $M_{\min}$, while for $V = \{0\}$, M_{ext}. If $U \subset V$, then $M_U \leq M_V$. Take $V = [0,a]$ to get a natural family.

Merging

A hazy simplicial set derived from an uber-metric space is a diagram (inverse system) of simplicial sets. From this, we construct a classical hazy simplicial set, which consists of a simplicial set (where $n-$dimensional simplices are identified) along with a haziness function. This hazy simplicial set is then realized as a metric space through a smearing process.

Merging

Consider a finite vertex set V, our data sample, on which a hazy simplicial set is built.

Merging

Properties of a hazy simplicial set enabling metric realization

1- Haziness of simplex $\geq \max$ haziness of faces.

2- Vertices have haziness $= 0$.

3- As scale $\geq$ diameter, entire space is represented as a full simplex equipped with a haziness function.

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Example: $\delta_1(x,y) = 1, \delta_1(x,z) = \delta_1(y,z) = 3$ and $\delta_2(x,z) = 1, \delta_2(x,y) = \delta_2(y,z) = 3$. With $M = M_{\min}$, $\delta_M(x,y) = 1, \delta_M(x,z) = 1, \delta_M(y,z) = 3$, violating triangle.

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We therefore need

$$d(x, y) := \inf_{x_o = x, x_1, \ldots, x_\ell = y} \sum_{i=1}^{\ell} \delta(x_{i-1}, x_i)\ . \qquad (13)$$

Star graphs

We had constructed star graphs centered at the sample points x, connecting them with their k nearest neighbors. But when x and y are both among the k nearest neighbors of each other, the weights of the edges in the corresponding star graphs may be different. And other weights are even infinite. So, we use the above scheme to merge these star graphs to reconstruct some graph that represents the entire sample.

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You may ask: Why don't we simply construct a graph from the sample by connecting sufficiently close points and use the original distances? Or connect two points when at least one of them is among the k nearest neighbors of the other?
One of the earlier schemes, **Isomap**, essentially does that, but the results become better when we use the apparently more complicated procedure suggested by **UMAP** and refined by us in **IsUMap**. This captures the local structure and the possibly varying density better. The local structure may look different from different points, and this is reflected in the modified star graphs.

Merging star graphs

All star graphs are defined on the same underlying set of sample points, and so we can identify points in different star graphs as $[x]$. When we use the minimum m-scheme, we get

$$d_\sim([x],[x']) = \inf(d_X(p_1,q_1) + \cdots + d_X(p_n,q_n)), \qquad (14)$$

where the infimum is taken over all pairs of sequences $(p_1, \cdots, p_n)$, $(q_1, \cdots, q_n)$ of elements of X, such that

$$p_1 \sim x, \quad q_n \sim x', \quad \text{and} \quad p_{i+1} \sim q_i \ \text{ for all } \ 1 \leq i \leq n-1, \qquad (15)$$

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Other m-schemes yield different results, and we can experiment which one works best for a given data sample.

IsUMap

Given a sample $x_1, \ldots, x_N$ from a metric data set

1. Construct star graphs centered at the x_i with uber-metrics (UM) that assign distance $= \infty$ between different points unless both are among x_i and its the k nearest neighbors. These metrics are normalized to adjust for non-uniform distributions of the samples and the curse of dimensionality.

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3. Use the Dijkstra algorithm to compute the distance function on the resulting global graph (Γ, d_Γ) .
4. Approximate it by a 2D graph (γ, d_γ) using multidimensional scaling, that is, minimizing a function like

$$\sum_{i,j=1,\ldots,N} (d_\gamma(y_i, y_j) - d_\Gamma(x_i, x_j))^2 \tag{16}$$

where $y_i \in \gamma$ corresponds to $x_i \in \Gamma$.

Hazy simplicial sets and TDA

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In a hazy simplicial complex, all simplices carry haziness values, and all the faces of σ must have values not larger than σ itself.
In short: The radius r of TDA that determines the intersections of distance balls becomes a haziness value.

Topology and curvature

Now, finally, the most important geometric concept, *curvature*, will enter.

Sectional curvature

The sectional curvatures, that is, the Riemannian curvatures evaluated on 2-dimensional tangent planes determine the metric locally completely. If they vanish, the manifold is flat, that is, locally Euclidean. The spaces of constant sectional curvatures (spheres for positive, hyperbolic spaces for negative curvature) are the basic model spaces of geometry.

Spaces of constant curvature serve as comparison spaces in geometry. When the curvature of a space satisfies $\lambda \leq K \leq \mu$, its geometry is between those of the spaces of constant curvature λ and μ.

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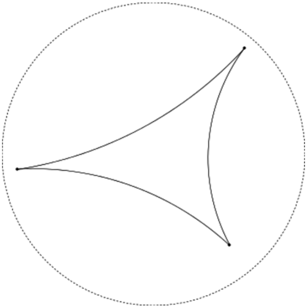
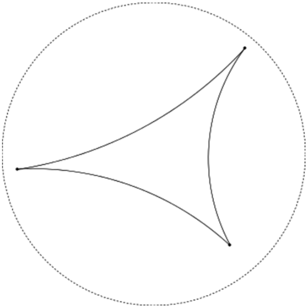
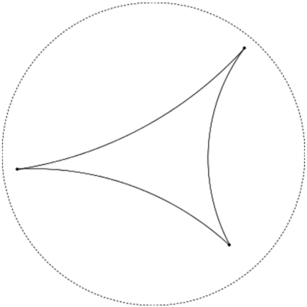
We can, for instance, compare the geometry of spaces of curvature ≤ 0 with that of Euclidean spaces.



A triangle in a space of negative curvature is thinner than in Euclidean space.



A triangle in a space of negative curvature is thinner than in Euclidean space. And it becomes the thinner, the more negative the curvature, and ultimately, when curvature $\to -\infty$, converges to a tripod.



A triangle in a space of negative curvature is thinner than in Euclidean space.
In fact, this is a local property that is characteristic for negative curvature. But such a property can also be checked in metric spaces more general than Riemannian manifolds. We only need geodesic triangles, that is, triangles whose sides are shortest geodesics.

It was then an important idea of Karl Menger (1902-1985), Abraham Wald (1902-1950), Alexandr Danilovic Alexandrov (1912-1999) and Herbert Busemann (1905-1994) to define curvature bounds for metric spaces more general than Riemannian manifolds in terms of inequalities for geodesic triangles.

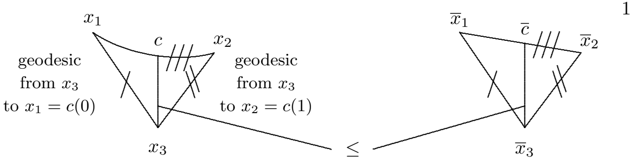


Figure 1: Comparison between a triangle in a space of nonpositive curvature in the sense of Alexandrov and the triangle with the same lengths of corresponding sides (indicated by slashes) in the Euclidean plane.

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But our data spaces are discrete, and therefore do not contain geodesic connections between data points. Therefore, we need a still more general concept.

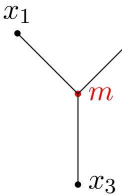
Sectional curvature

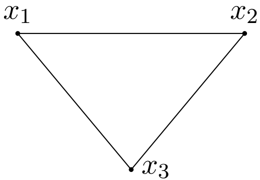
With Parvaneh Joharinad, I have developed a notion of sectional curvature that applies to general metric spaces and can be applied in data analysis.

Here is the basic question:

Sectional curvature

How to distinguish a tripod from a triangle?





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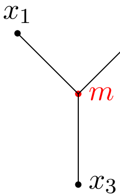

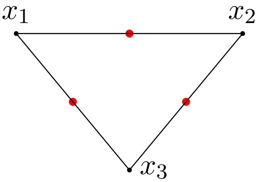

Let $d(x_i, x_j) = 2r$ for $i \neq j$. Then the balls $B(x_i, r) = \{x : d(x_i, x) \leq r\}$ intersect pairwise in each graph.

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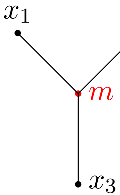

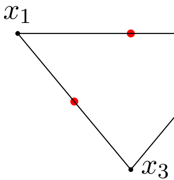

Let $d(x_i, x_j) = 2r$ for $i \neq j$. Then the balls $B(x_i, r) = \{x : d(x_i, x) \leq r\}$ intersect pairwise in each graph. But in the tripod, they also have a triple intersection

$$B(x_1, r) \cap B(x_2, r) \cap B(x_3, r) = \{m\} \neq \emptyset$$

whereas in the triangle, only

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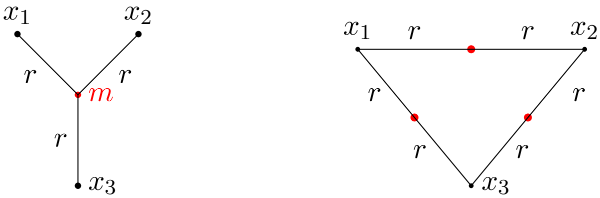
Sectional curvature



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These are extreme cases, and when $B(x_i, r) \cap B(x_j, r) \neq \emptyset$, there exists some $1 \leq \lambda \leq 2$ with

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quantifies the curvature. The smaller λ, the more negative the curvature is.

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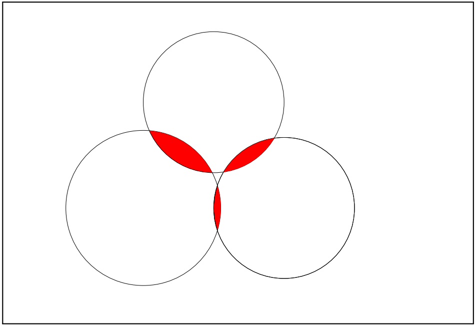
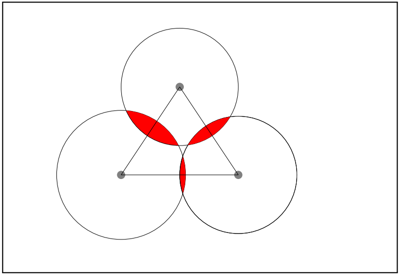


Topological Data Analysis (TDA): For metric family $(x_i)_{i \in I}, d$ and $r > 0$, the Čech complex contains a q-simplex whenever

$$\bigcap_{i=1,\dots,q+1} B(x_i, r) \neq \emptyset.$$

Its homology (unfilled simplices) varies as a function of r. Thus, the vertices (0-simplices) of our simplicial complex correspond to the balls. Two vertices are connected by an edge (1-simplex) when the balls intersect, and we fill a triangle (2-simplex) when the three balls have a common intersection.



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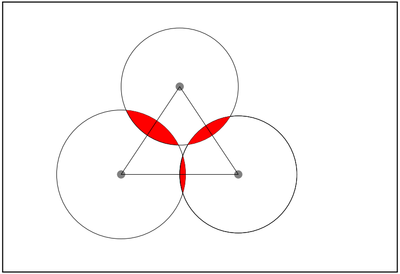

Here, triangle is not filled, because no triple intersection. If the balls were smaller, they would not intersect pairwise, and if they were larger, we would also have a triple intersection. Thus, here we have a scale with non-trivial homology.

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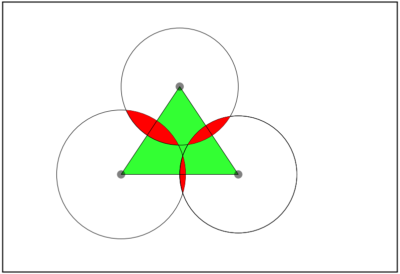
Its homology (unfilled simplices) varies as a function of r.
In contrast, in the Vietoris-Rips complex, simplices are filled whenever the balls around their vertices intersect pairwise.



Our notion of sectional curvature quantifies the difference between the Čech complex and the Vietoris-Rips complex.

How much does one have to enlarge balls that intersect pairwise to get triple intersections?

Our notion of sectional curvature quantifies the difference between the Čech complex and the Vietoris-Rips complex.

How much does one have to enlarge balls that intersect pairwise to get triple intersections?
The less one has to enlarge them, the more negative the curvature.
Thus, we can understand TDA from a geometric perspective.

The indicated notion of sectional curvature for metric spaces[8] is useful for the large scale analysis of metric spaces. When the space is discrete, one may allow for some $\delta > 0$, depending on the scale of the metric and ask for intersections of radius $r + \delta$.

[8] P.Joharinad, J.J. *Mathematical principles of topological and geometric data analysis*, Math of Data, Springer, 2023

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For instance, for a traffic network, it is useful to distinguish a large scale tripod type pattern, with a center to which all lines go, like the French railway system with Paris as the center, from a ring type structure where locations typically lie on a cycle.

[8] P.Joharinad, J.J. *Mathematical principles of topological and geometric data analysis*, Math of Data, Springer, 2023

Curvature and convexity

The more negative the curvature, the more convex the space becomes.[9]

A region R is *convex* if whenever x and y are in R, then also any point between x and y is in R as well.

[9] J.J., *Nonpositive curvature*, Birkhäuser, 1997

The geometry of conceptual spaces

- *Adjectives* are represented by *convex* regions in conceptual spaces (P.Gärdenfors, *The Geometry of Meaning*, MIT Press, 2014).
  Clustering algorithms also typically, but not necessarily, yield convex regions.

The geometry of conceptual spaces

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  In particular, the regions for the various colors (green, red, blue, yellow,...) in color space are convex.

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