Duality for Partial Boolean Algebras

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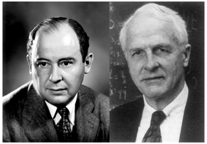
The essence of contextuality

- Not all properties may be observed simultaneously.
- Sets of jointly observable properties provide **partial, classical snapshots**.
- Contextuality arises where there is a family of data which is

locally consistent but globally inconsistent

Background: traditional quantum logic



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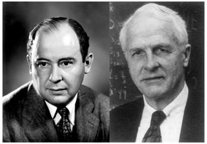

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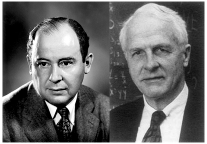

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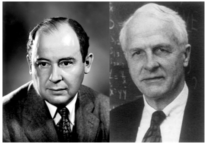

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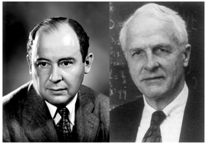

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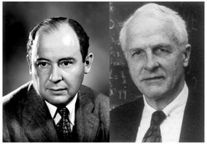

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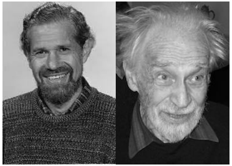
- Interpret $\wedge$ (infimum) and $\vee$ (supremum) as logical operations.
- Distributivity fails: $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$.
- Only commuting measurements can be performed together.
  So, what is the operational meaning of $p \wedge q$, when p and q **do not commute**?

Quantum physics and logic



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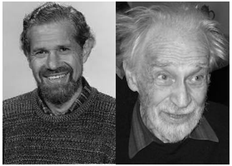

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- Only admit physically meaningful operations.
- Represent incompatibility by **partiality**.

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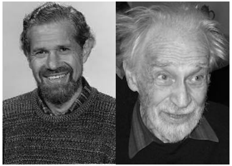

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Kochen (2015), '*A reconstruction of quantum mechanics*'.

- Kochen develops a large part of foundations of quantum theory in this framework.

Partial Boolean algebras

Partial Boolean algebra $\langle A, \odot, 0, 1, \neg, \vee, \wedge \rangle$:

- a set A
- a reflexive, symmetric binary relation $\odot$ on A, read *commeasurability* or *compatibility*
- constants $0, 1 \in A$
- (total) unary operation $\neg : A \longrightarrow A$
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Morphisms of pBAs are maps preserving commeasurability, and the operations wherever defined. This gives the category **pBA**.

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Kochen & Specker (1965).

Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} \geq 3$, and $\mathrm{P}(\mathcal{H})$ its pBA of projectors.

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- No assignment of truth values to all propositions that respects the logical operations on jointly testable propositions.
- Spectrum of a pBA cannot have *points*...

Conditions of impossible experience

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Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S (i.e. no homomorphism to* **2***)*
2. *For some* ***propositional contradiction*** $\varphi(\vec{x})$ *and assignment* $\vec{x} \mapsto \vec{a}$*,*

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How can the world be this way? Still an ongoing debate, an enduring mystery …

Contrast with Intuitionistic logic

Say that a **classical contradiction** is a propositional formula φ such that $\mathrm{CL} \vdash \neg\varphi$.

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If $\mathrm{CL} \vdash \neg\varphi$*, then* $\mathrm{IL} \vdash \neg\varphi$.

Proof.

If $\mathrm{CL} \vdash \neg\varphi$, then by Glivenko's theorem, $\mathrm{IL} \vdash \neg\neg\neg\varphi$. Since $\mathrm{IL} \vdash \neg\neg\neg p \longrightarrow \neg p$, it follows that $\mathrm{IL} \vdash \neg\varphi$. □

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As a corollary, we obtain:

Theorem

A classical contradiction cannot be satisfied in any sound semantics for intuitionistic logic.

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Is there a “logical” proof?

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Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

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By contrast, coequalisers, and general colimits, are shown to exist by Heunen and van der Berg by an appeal to the Adjoint Functor Theorem. One of our contributions is to give an explicit construction of the needed colimits,.

More generally, we use this approach to prove the following result, which freely generates from a given partial Boolean algebra a new one where prescribed additional commeasurability relations are enforced between its elements.

Theorem

Given a partial Boolean algebra A and a binary relation $\circledcirc$ on A, there is a partial Boolean algebra $A[\circledcirc]$ such that:

- *There is a* **pBA**-*morphism* $\eta : A \longrightarrow A[\circledcirc]$ *such that* $a \circledcirc b \;\Rightarrow\; \eta(a) \odot_{A[\circledcirc]} \eta(b)$.
- *For every partial Boolean algebra B and* **pBA**-*morphism* $h : A \longrightarrow B$ *such that* $a \circledcirc b \;\Rightarrow\; h(a) \odot_B h(b)$, *there is a unique homomorphism* $\hat{h} : A[\circledcirc] \longrightarrow B$ *such that*

$$\begin{array}{ccc} A & \xrightarrow{\;\eta\;} & A[\circledcirc] \\ & \searrow^{h} & \downarrow{\hat{h}} \\ & & B \end{array}$$

This result is proved constructively, by giving proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A. (In fact, we start with a set of "pre-terms", and also give rules for definedness).

The inductive construction

$$\frac{a \in A}{\imath(a)\downarrow} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

$$\frac{}{0 \equiv \imath(0_A),\ 1 \equiv \imath(1_A),\ \neg\imath(a) \equiv \imath(\neg_A a)}$$

$$\frac{a \odot_A b}{\imath(a) \wedge \imath(b) \equiv \imath(a \wedge_A b),\ \imath(a) \vee \imath(b) \equiv \imath(a \vee_A b)}$$

$$\frac{}{0\downarrow,\ 1\downarrow} \qquad \frac{t \odot u}{t \wedge u\downarrow,\ t \vee u\downarrow} \qquad \frac{t\downarrow}{\neg t\downarrow}$$

$$\frac{t\downarrow}{t \odot t,\ t \odot 0,\ t \odot 1} \qquad \frac{t \odot u}{u \odot t} \qquad \frac{t \odot u,\ t \odot v,\ u \odot v}{t \wedge u \odot v,\ t \vee u \odot v} \qquad \frac{t \odot u}{\neg t \odot u}$$

$$\frac{t\downarrow}{t \equiv t} \qquad \frac{t \equiv u}{u \equiv v} \qquad \frac{t \equiv u,\ u \equiv v}{t \equiv v} \qquad \frac{t \equiv u,\ u \odot v}{t \odot v}$$

$$\frac{\varphi(\vec{x}) \equiv_{\mathrm{Bool}} \psi(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{\varphi(\vec{v}) \equiv \psi(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \circledcirc a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\circledcirc, \equiv]$.

Theorem

Let $h : A \longrightarrow B$ be a **pBA***-morphism such that $a \circledcirc a' \;\Rightarrow\; h(a) = h(a')$. Then there is a unique* **pBA***-morphism $\hat{h} : A[\circledcirc, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This result can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

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As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

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In the case of a partial Boolean algebra with the K-S property of not having a homomorphism to **2**, the colimit of its diagram of boolean subalgebras must be **1**.

KS-property and colimits

We can turn this into a theorem:

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We could say that such a diagram is “implicitly contradictory”, and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

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2. *The colimit of the diagram of boolean subalgebras of A in* **BA** *is* **1**.

In fact, we can formulate the K-S property directly for diagrams of Boolean algebras, without referring to partial boolean algebras at all.

We say that a diagram in **BA** is K-S if its colimit in **BA** is **1**.

We could say that such a diagram is "implicitly contradictory", and in trying to combine all the information in a colimit, we obtain the manifestly contradictory **1**.

A partial Boolean algebra with the K-S property – such as $P(\mathcal{H})$ – holds this implicitly contradictory information together in a single structure.

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Firstly, all elements are commeasurable in $A[A^2]$, so it is a Boolean algebra. Moreover, there is a morphism $\eta : A \longrightarrow A[A^2]$. Thus if A is K-S, we must have $A[A^2] = \mathbf{1}$.

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Conversely, suppose that $A[A^2] = \mathbf{1}$, and there is a morphism $A \longrightarrow B$ to a Boolean algebra A. By the universal property of $A[A^2]$, there is a morphism $A[A^2] \longrightarrow B$, and since $A[A^2] = \mathbf{1}$, we must have $B = \mathbf{1}$. Thus A is K-S. □

Tensor product and the emergence of non-classicality

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One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which **does not** have the K–S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which **does**.

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Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** *such that the functor* $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ *is* ***strong monoidal*** *with respect to this structure, i.e. such that* $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$*?*

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A positive answer to this question would offer a complete logical characterisation of the Hilbert space tensor product, and provide an important step towards giving logical foundations for quantum theory in a form useful for quantum information and computation.

Tensor products of partial Boolean algebras

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In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

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Our Theorem 5 allows us to give an explicit description of this construction using generators and relations.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

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There is a lax monoidal functor $\mathbf{P} : \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K}) \longrightarrow \mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathrm{P}(\mathcal{H})$ and $\mathrm{P}(\mathcal{K})$ into $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

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It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

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Interestingly, in (Kochen 2015) it is shown that the images of $\mathrm{P}(\mathcal{H})$ and $\mathrm{P}(\mathcal{K})$, for any finite-dimensional $\mathcal{H}$ and $\mathcal{K}$, generate $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$. This is used there to justify the claim contradicted by the previous paragraph. The gap in the argument is that more relations hold in $\mathrm{P}(\mathcal{H} \otimes \mathcal{K})$ than in $\mathrm{P}(\mathcal{H}) \otimes \mathrm{P}(\mathcal{K})$.

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Nevertheless, this result is very suggestive. It poses the challenge of finding a stronger notion of tensor product.

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An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t\downarrow$ can be derived, then $u\downarrow$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

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To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

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However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not.

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Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

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Definition

A partial Boolean algebra A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.
We write **epBA** for the full subcategory of **pBA** whose objects are partial Boolean algebras satisfying LEP.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

Definition

A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

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As an immediate consequence, any $\mathrm{P}(\mathcal{H})$ satisfies LEP.

A reflective adjunction for logical exclusivity

We can of course form the partial Boolean algebra $A[\perp]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

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We can of course form the partial Boolean algebra $A[\bot]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\bot]$, it may fail to hold for other elements in $A[\bot]$.

However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

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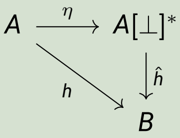
However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

This LEP-isation is analogous to e.g. the way one can 'abelianise' any group, or use Stone–Čech compactification to form a compact Hausdorff space from any topological space.

Theorem

The category **epBA** *is a reflective subcategory of* **pBA***, i.e. the inclusion functor* I : **epBA** $\longrightarrow$ **pBA** *has a left adjoint* X : **pBA** $\longrightarrow$ **epBA**. *Concretely, to any partial Boolean algebra* A*, we can associate a Boolean algebra* $X(A) = A[\perp]^*$ *which satisfies LEP such that:*

- *there is a homomorphism* $\eta : A \longrightarrow A[\perp]^*$;
- *for any homomorphism* $h : A \longrightarrow B$ *where* B *is a partial Boolean algebra* B *satisfying LEP, there is a unique homomorphism* $\hat{h} : A[\perp]^* \longrightarrow B$ *such that:*



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$$
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\end{array}
$$

The proof of this result follows from a simple adaptation of the proof of Theorem 5, namely adding the following rule to the inductive system presented in Table 1:

$$
\frac{u \wedge t \equiv u, \;\; v \wedge \neg t \equiv v}{u \odot v}
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How close does it it get us to the full Hilbert space tensor product?

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Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

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Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A is K-S if and only if $A[R]$ is K-S.

Proof.

If A is not K-S, it has a homomorphism to a non-trivial Boolean algebra B. By the universal property of $A[R]$, there is a homomorphism $\hat{h} : A[R] \longrightarrow B$. Thus $A[R]$ is not K-S. Conversely, if there is a morphism $k : A[R] \longrightarrow B$ to a non-trivial Boolean algebra B, then $k \circ \eta : A \longrightarrow B$, so A is not K-S. □

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If A and B are not K-S, then neither is $A \otimes B[\bot]^k$.

Proof.

If A and B are not K-S, they have homomorphisms to **2**, and hence so does $A \oplus B$. Applying the previous theorem inductively $k+1$ times, so does $A \otimes B[\bot]^k = A \oplus B[\oplus][\bot]^k$. □

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Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a K-S paradox if none was present if A or B.

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If A and B are not K-S, they have homomorphisms to $\mathbf{2}$, and hence so does $A \oplus B$. Applying the previous theorem inductively $k+1$ times, so does $A \otimes B[\perp]^k = A \oplus B[\oplus][\perp]^k$. □

Under the conjecture that $A[\perp]^*$ coincides with iterating $A[\perp]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a K-S paradox if none was present if A or B.

So we have narrowed, but not closed the gap ...

Duality for partial Boolean Algebras?

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At first sight, this looks hopeless:

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We will instead generalize the **Tarski duality** for complete atomic Boolean algebras (CABAs)

CABAs

Definition (Complete Boolean algebra)

A Boolean algebra A is said to be **complete** if any subset of elements $S \subseteq A$ has a supremum $\bigvee S$ in A (and consequently an infimum $\bigwedge S$, too). It thus has additional operations

$$\bigwedge, \bigvee : \mathcal{P}(A) \longrightarrow A\ .$$

Definition (Atomic Boolean algebra)

An **atom** of a Boolean algebra is a minimal non-zero element, i.e. an element $x \neq 0$ such that $a \leq x$ implies $a = 0$ or $a = x$.

Atoms are “state descriptions” or “possible worlds”.

A Boolean algebra A is called **atomic** if every non-zero element sits above an atom, i.e. for all $a \in A$ with $a \neq 0$ there is an atom x with $x \leq a$.

A **CABA** is a complete, atomic Boolean algebra.

Tarski duality

$$\mathbf{CABA} \underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}} \mathbf{Set}^{\mathrm{op}} \qquad \mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}}$$

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$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{CABA}$ is the contravariant powerset functor:

- on objects: a set X is mapped to its powerset $\mathcal{P}X$ (a CABA).
- on morphisms: a function $f : X \longrightarrow Y$ yields a complete Boolean algebra homomorphism

$$\begin{aligned} \mathcal{P}(f) : \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X) \\ (T \subseteq Y) &\longmapsto f^{-1}(T) = \{x \in X \mid f(x) \in T\} \end{aligned}$$

Tarski duality

$$\mathbf{CABA} \;\underset{\mathrm{At}}{\overset{\mathcal{P}}{\rightleftarrows}}\; \mathbf{Set}^{\mathrm{op}} \qquad (\mathbf{CABA} \cong \mathbf{Set}^{\mathrm{op}})$$

$\mathbf{At} : \mathbf{CABA}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ is defined as follows:

- on objects: a CABA A is mapped to its set of atoms.
- on morphisms: a complete Boolean homomorphism $h : A \longrightarrow B$ yields a function

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mapping an atom y of B to the unique atom x of A such that $y \leq h(x)$.

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Duality for partial CABAs

Partial CABAs

Definition (partial complete BA)

A **partial complete Boolean algebra** is a pBA with an additional (partial) operation

$$\bigvee : \odot \longrightarrow A$$

satisfying the following property: any set $S \in \odot$ is contained in a set $T \in \odot$ which forms a complete Boolean algebra under the restriction of the operations.

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Note that $\mathrm{P}(\mathcal{H})$ is a partial CABA. Atoms are the rank-1 projectors (one-dimensional subspaces), i.e. the **pure states**.

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Duality for partial CABAs: the idea

- The key idea is to replace **sets** by certain **graphs**.
- Adjacency generalizes $\neq$, thus sets embed as **complete graphs**.
- These exclusivity graphs are the “non-commutative spaces” in this duality.
- Morphism of graphs are certain relations, generalizing the functional relations which appear in classical Tarski duality.

Graph theory notions

Definition

A **graph** $(X, \#)$ is a set equipped with a symmetric irreflexive relation.

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Given a vertex $x \in X$ and sets of vertices $S, T \subset X$, we write:

- $x \# S$ when for all $y \in S$, $x \# y$;
- $S \# T$ when for all $x \in S$ and $y \in T$, $x \# y$;
- $x^{\#} := \{y \in X \mid y \# x\}$ for the neighbourhood of the vertex x;
- $S^{\#} := \bigcap_{x \in S} x^{\#} = \{y \in X \mid y \# S\}$ for the common neighbourhood of the set S.

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A **clique** is a set of pairwise-adjacent vertices, i.e. a set $K \subset X$ with $x \# K \setminus \{x\}$ for all $x \in K$.

A graph $(X, \#)$ has **finite clique cardinal** if all cliques are finite sets.

Graph of atoms

Definition (Graph of atoms)

The **graph of atoms** of a partial Boolean algebra A, denoted $\mathrm{At}(A)$, has as vertices the atoms of A and an edge between atoms x and x' if and only if $x \odot x'$ and $x \wedge x' = 0$.

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Recall that in a CABA, any element is uniquely written as a join of atoms, viz. $a = \bigvee U_a$ with

$$U_a := \{x \in \mathrm{At}(A) \mid x \leq a\}$$

In a pBA, U_a may not be pairwise commeasurable, hence their join need not even be defined.

Elements from atoms

Proposition

Let A be a transitive partial CABA. For any element $a \in A$, it holds that $a = \bigvee K$ for any clique K of $\mathrm{At}(A)$ which is maximal in U_a.

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Given two maximal cliques K and L, this yields an equality

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The key to reconstructing a partial CABA from its atoms lies in characterising such equalities,

Proposition

Let K and L be cliques in $\mathrm{At}(A)$. Then $\bigvee K = \bigvee L$ iff $K^{\#} = L^{\#}$.

Partial CABA from its graph of atoms

Writing

$$K \equiv L \;:\Leftrightarrow\; K^{\#} = L^{\#},$$

elements of A are in 1-to-1 correspondence with $\equiv$-equivalence classes of cliques of $\mathrm{At}(A)$.

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- $[K] \vee [L] = [K' \cup L']$.
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Which conditions on a graph $(X, \#)$ allow for such reconstruction?

Complete exclusivity graphs

Definition

A **complete exclusivity graph** is a graph $(X, \#)$ such that for K, L cliques and $x, y \in X$:

1. If $K \sqcup L$ is a maximal clique, then $K^\# \# L^\#$, i.e. $x \# K$ and $y \# L$ implies $x \# y$.
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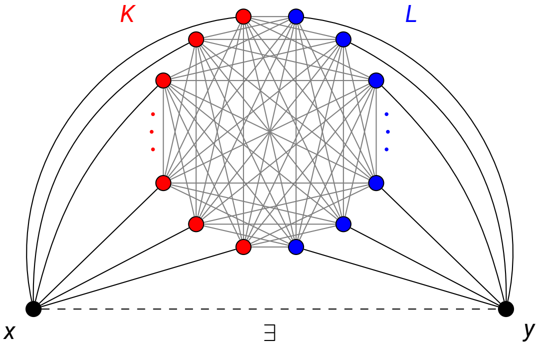

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A helpful intuition is to see these as generalising sets with a $\neq$ relation (the complete graph).

- A graph is symmetric and irreflexive.
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- A graph is symmetric and irreflexive.
- To be an inequivalence relation, we need cotransitivity: $x \# z$ implies $x \# y$ or $y \# z$.
- Condition 1. is a weaker version of cotransitivity.
- Condition 2. eliminates redundant elements: cotransitive + 2. implies $\neq$.

Graph of atoms is complete exclusivity graph

Proposition

Let A be a partial Boolean algebra. Then $\mathrm{At}(A)$ is a complete exclusivity graph.

Proof.

Let $K, L \subset X$ such that $K \sqcup L$ is a maximal clique, and let x, y be atoms of A.
$c := \bigvee K = \neg \bigvee L$.
$x \# K$ means $x \leq \neg \bigvee K = \neg c$ and $x \# L$ means $y \leq \neg \bigvee L = c$.
By transitivity, we conclude that $x \odot y$, □

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Morphisms of complete exclusivity graphs

What about morphisms?

Definition

A morphism $(X, \#) \longrightarrow (Y, \#)$ is a relation $R : X \longrightarrow Y$ satisfying:

1. $x \, R \, y$, $x' \, R \, y'$, and $y \# y'$ implies $x \# x'$
2. if K is a maximal clique in Y, $R^{-1}(K)$ contains a maximal clique.
3. for each $y \in Y$, $(R^{-1}(\{y\}))^{\#\#} = R^{-1}(\{y\})$.

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Given $h : A \longrightarrow B$ define $y \; R \; x$ iff $y \leq h(x)$.

Morphisms of CE graphs and pCABA homomorphisms

Proposition

Let A and B be transitive partial CABAs. Given $h : A \longrightarrow B$ a partial complete Boolean algebra homomorphism, the relation $R_h : \mathrm{At}(B) \longrightarrow \mathrm{At}(A)$ given by

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Proposition

For any A and B be transitive partial CABAs, $\mathbf{epCABA}(A, B) \cong \mathbf{XGph}(\mathrm{At}(B), \mathrm{At}(A))$.

Global points

Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

Global points

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i.e. a subset of atoms of A satisfying:

1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

Global points

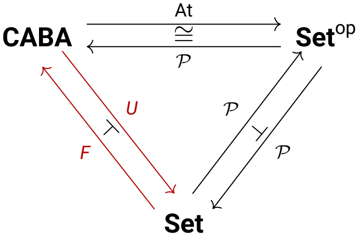
Homomorphism $A \longrightarrow 2$ corresponds to morphism $K_1 \longrightarrow \mathrm{At}(A)$,

i.e. a subset of atoms of A satisfying:

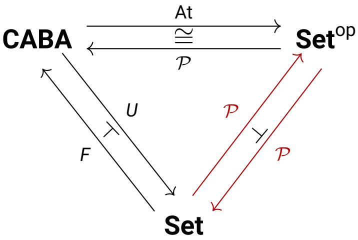
1. it is an independent (or stable) set
2. it is a maximal clique transversal, i.e. it has a vertex in each maximal clique

The extensive literature on Kochen-Specker constructions is concerned with building graphs which have no such transversals, thus showing that the corresponding pBA's have no points.

Free-forgetful adjunction for CABAs

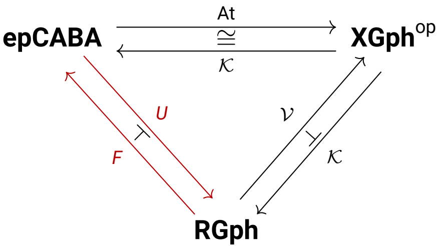


Free-forgetful adjunction for CABAs

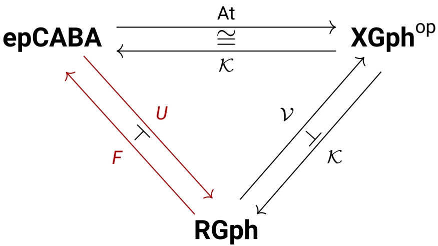


- Under the duality, it corresponds to the contravariant powerset self-adjunction.
- It gives the construction of the free CABA as a double powerset.

Free-forgetful adjunction for partial CABAs

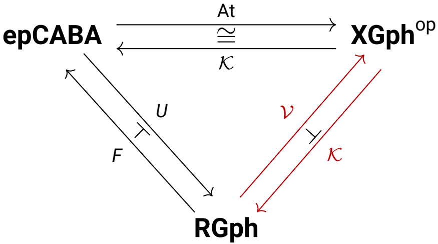


Free-forgetful adjunction for partial CABAs



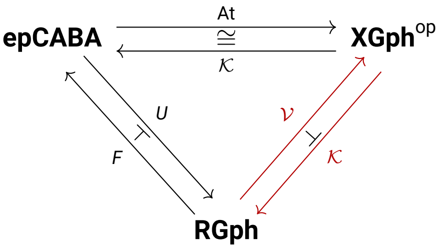
- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$

Free-forgetful adjunction for partial CABAs



- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA.

Free-forgetful adjunction for partial CABAs



- Universe of a pCABA is a reflexive (compability) graph $\langle A, \odot \rangle$
- Under duality it corresponds to adjunction between **compatibility** and **exclusivity** graphs.
- This gives a concrete construction of the free CABA. A compatibility $\langle P, \odot \rangle$ to a graph with vertices $\langle C, \gamma : C \longrightarrow \{0,1\} \rangle$ where C maximal compatible set, and edges

$$\langle C, \gamma \rangle \# \langle D, \delta \rangle \qquad \text{iff} \qquad \exists x \in C \cap D.\ \ \gamma(x) \neq \delta(x)\ .$$